Effective Constraints

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Constrained systems

- Classically constraints restrict "physically accessible" region of the phase space Γ_{class}; functions $C : \Gamma_{\text{class}} \to \mathbb{R}$, set $C = 0$.
- Arise naturally from action principle; indicate presence of gauge degrees of freedom.
- Constraints may be solved before quantization, but in some cases gauge freedom plays a crucial role.
- Dirac's prescription: $\hat{C}\psi_{\text{phys}} = 0$, clear if $\psi_{\text{phys}} \in \mathscr{H}_{\text{kin}}$.
- Otherwise construct $\mathscr{H}_{\text{phys}}$ equipped with $\langle , \rangle_{\text{phys}}$—non-trivial.
- Is there a simpler way to get quantum corrections?

Main idea

- Supplement Γ_{class} with leading order "quantum parameters" and associated constraints—should be easier than constructing $\mathscr{H}_{\text{phys}}$.
- "Quantum parameters"—some functions of expectation values; e.g. $\langle \hat{O}^n \rangle - \langle \hat{O} \rangle^n \neq 0$ is a departure from classical behavior.
- Inspiration—geometrical QM where $\langle \hat{O} \rangle$-s are functions on a symplectic manifold. [e.g. A. Ashtekar, T. Schilling 1997]
- Here focusing on systems with finite-dimensional Γ_{class} we:
  - formulate suitable analogue of $\hat{C}\psi_{\text{phys}} = 0$ on $\langle \hat{O} \rangle$-s
  - through an example show how these lead to quantum corrections.

Basic assumptions

- Take a quantum system that is well understood in the absence of constraints.
- In particular assume that
  - observables of interest form some (known) associative algebra: for each pair $\mathbf{a}, \mathbf{b} \in \mathscr{A}$ there is $(\mathbf{ab}) \in \mathscr{A}$
  - commutator algebra $[\mathbf{a}, \mathbf{b}] = \mathbf{ab} - \mathbf{ba}$ is the quantum version of the classical Poisson algebra
  - there is a single constraint $\mathbf{C} \in \mathscr{A}$ (some polynomial in observables with some ordering chosen).
- Instead of vector-states look at complex linear maps $\alpha \in L(\mathscr{A} \to \mathbb{C})$ satisfying $\alpha(\mathbb{1}) = 1$. (E.g. $\rho(\hat{O}) = Tr(\hat{O}\hat{\rho})$.)

Constraint functions

- First note the natural linear left and right action of $\mathscr{A}$ on the maps
  - left: $(\mathbf{a}\alpha)(\mathbf{b}) = \alpha(\mathbf{ab})$
  - right: $(\alpha\mathbf{a})(\mathbf{b}) = \alpha(\mathbf{ba})$
- Substitute Dirac's condition by one of the following
  - $\mathbf{C}\alpha = 0$ implying $\alpha(\mathbf{Ca}) = 0 \ \forall \mathbf{a} \in \mathscr{A}$
  - $\alpha\mathbf{C} = 0$ implying $\alpha(\mathbf{aC}) = 0 \ \forall \mathbf{a} \in \mathscr{A}$
- In many cases possible to satisfy one, but not both e.g.:
  - $\hat{x}, \hat{p} \in \mathscr{A}$ subject to $[\hat{x}, \hat{p}] = i\hbar$ take $\mathbf{C} = \hat{x}$
  - demand $\hat{x}\alpha = 0$ in particular $\alpha(\hat{x}\hat{p}) = 0$
  - then $(\alpha\hat{x})(\hat{p}) = \alpha(\hat{p}\hat{x}) = \alpha(\hat{x}\hat{p} - [\hat{x}, \hat{p}]) = -i\hbar$, hence $\alpha\hat{x} \neq 0$
- Forced to use complex maps!
- Here we pick $\alpha\mathbf{C} = 0$; potentially an infinite number of conditions.
  (Additional structure may reduce this number dramatically [A. Corichi 2008])

Geometry

- The set of normalized linear maps $L(\mathscr{A} \to \mathbb{C})$ forms a complex affine space and hence a (complex) differential manifold—denote Γ.
- Each $\mathbf{a} \in \mathscr{A}$ assigns a function on Γ: $\langle \mathbf{a} \rangle(\alpha) = \alpha(\mathbf{a})$ (henceforth simply $\langle \mathbf{a} \rangle$)
    - this assignment is linear $\langle \mathbf{a} + \mathbf{b} \rangle = \langle \mathbf{a} \rangle + \langle \mathbf{b} \rangle$
    - Each $\alpha \in \Gamma$ is entirely defined by the values of $\langle \mathbf{a} \rangle$-s
    - a linear basis $\{\mathbf{e}_i\}$ of $\mathscr{A}$ gives a set of coordinate functions $\langle \mathbf{e}_i \rangle$
- Γ is equipped with a natural Poisson structure defined by the algebra commutator: $\{\langle \mathbf{a} \rangle, \langle \mathbf{b} \rangle\} := \frac{1}{i\hbar}\langle [\mathbf{a}, \mathbf{b}] \rangle$ (extend using Leibnitz rule)
- Poisson vector fields formally generate invertible (algebra preserving) transformations: $\{\langle \mathbf{a} \rangle, \langle \mathbf{b} \rangle\} = \frac{d}{d\epsilon}\langle \exp(-\frac{\epsilon}{i\hbar}\mathbf{a})\mathbf{b}\exp(\frac{\epsilon}{i\hbar}\mathbf{a}) \rangle \mid_{\epsilon=0}$

Gauge flows

- The set of constraint functions $\langle \mathbf{aC} \rangle = 0$ defines a smooth submanifold $\Sigma \subset \Gamma$. (Maps satisfying $\alpha \mathbf{C} = 0$ form an affine subspace.)
- The constraints are closed under the Poisson bracket (1st class). The associated flows are tangent to Σ (i.e. constraint preserving).
- These are analogues of classical gauge flows—true degrees of freedom are given by gauge invariant functions on Σ.
- A constrained Poisson manifold and a constrained symplectic manifold can be analyzed analogously.
- In particular $\Sigma/(\text{gauge orbits})$ naturally inherits Poisson structure from Γ.
- A truncation, if needed, should leave us with a 1st class system on a Poisson manifold.

Constraints on free Newtonian particle

- The following procedure should apply to any polynomial constraint in $\mathscr{A}$ generated by a finite-dimensional Lie algebra.
- Free system: two canonical pairs $\{\mathbf{q}, \mathbf{p}; \mathbf{t}, \mathbf{p}_t\}$
- Let $\mathscr{A}$ consist of identity and all ordered polynomials in the canonical variables subject to the canonical commutation relations.
- A point on $\Gamma \cong L(\mathscr{A} \to \mathbb{C})$ is completely determined by the values it assigns to polynomials $\mathbf{q}^k\mathbf{p}^l\mathbf{t}^m\mathbf{p}_t^n$ (reordering adds lower order terms)
- Introduce constraint: $\boxed{\mathbf{C} = \mathbf{p}_t + \frac{\mathbf{p}^2}{2M}}$. Classically—a time-deparameterized version of a free non-relativistic particle.
- Systematically impose constraints order by order: $C_{\mathbf{q}^k\mathbf{p}^l\mathbf{t}^m\mathbf{p}_t^n} = \langle \mathbf{q}^k\mathbf{p}^l\mathbf{t}^m\mathbf{p}_t^n\mathbf{C} \rangle = 0$ —infinitely many.

Semiclassical reduction

- No approximations used yet—for practical calculations need finite number of equations. Here expand about classical limit.
- To add leading order "quantum parameters" to classical phase space, need functions on Γ that measure "quantumness".
- Moments of observables are non-linear functions on Γ that have a clear notion of order: $\langle (\mathbf{q}-\langle\mathbf{q}\rangle)^k(\mathbf{p}-\langle\mathbf{p}\rangle)^l(\mathbf{t}-\langle\mathbf{t}\rangle)^m(\mathbf{p}_t-\langle\mathbf{p}_t\rangle)^n\rangle_{\mathrm{Weyl}}$ for semiclassical states $\propto \hbar^{\frac{1}{2}(k+l+m+n)}$
- Reduce system of constraints and the Poisson structure:
  1. recast equations in terms of moments
  2. assign appropriate powers of $\hbar^{\frac{1}{2}}$
  3. drop all terms of order above N
- This reduction results in a finite number of non-trivial first-class constraints and a closed Poisson structure to order N.

Corrections up to 2nd order

- Degrees of freedom: 4 expectation values $a = \langle \mathbf{a} \rangle$; 4 spreads $(\Delta a)^2 = \langle (\mathbf{a} - a)^2 \rangle$ and 6 variances $\Delta(ab) = \langle (\mathbf{a} - a)(\mathbf{b} - b) \rangle_{\mathrm{Weyl}}$
- 5 non-trivial constraints left:

$$\langle \mathbf{C} \rangle = p_t + \frac{p^2}{2M} + \frac{(\Delta p)^2}{2M} = 0; \quad \langle \mathbf{pC} \rangle = \Delta(pp_t) + \frac{p(\Delta p)^2}{M} = 0; \quad \langle \mathbf{p}_t\mathbf{C} \rangle = (\Delta p_t)^2 + \frac{p\Delta(pp_t)}{M} = 0;$$
$$\langle \mathbf{qC} \rangle = \Delta(qp_t) + \frac{i\hbar p}{2M} + \frac{p\Delta(qp)}{M} = 0; \quad \langle \mathbf{tC} \rangle = \frac{p\Delta(pt)}{M} + \Delta(tp_t) + \frac{i\hbar}{2} = 0.$$

- Eliminating $p_t, (\Delta p_t)^2, \Delta(pp_t), \Delta(qp_t), \Delta(tp_t)$ we can write the gauge invariant functions on Σ as:

$$\mathscr{P} = p; \qquad \mathscr{Q} = q - \frac{tp}{M} - \frac{\Delta(tp)}{M}; \quad \Delta(\mathscr{Q}\mathscr{P}) = \Delta(qp) - \Delta(tp) - \frac{t(\Delta p)^2}{M};$$
$$(\Delta\mathscr{P})^2 = (\Delta p)^2; \quad (\Delta\mathscr{Q})^2 = (\Delta q)^2 - \frac{2p\Delta(qt)}{M} + \frac{p^2(\Delta t)^2}{M^2} + \frac{t^2(\Delta p)^2}{M^2} - \frac{2t}{M}\left(\Delta(qp) - \Delta(tp)\right).$$

- Poisson algebra as expected for 1 canonical pair: $\{\mathscr{Q}, \mathscr{P}\} = 1;$

$$\{(\Delta\mathscr{Q})^2, (\Delta\mathscr{P})^2\} = 4\Delta(\mathscr{Q}\mathscr{P}); \quad \{(\Delta\mathscr{Q})^2, \Delta(\mathscr{Q}\mathscr{P})\} = 2(\Delta\mathscr{Q})^2; \quad \{(\Delta\mathscr{P})^2, \Delta(\mathscr{Q}\mathscr{P})\} = -2(\Delta\mathscr{P})^2.$$

Gauge fixing

- To identify gauge-invariant variables as observables —enforce relations that would be satisfied by $\langle , \rangle_{\rm phys}$
  - reality: $\mathscr{P}, \mathscr{Q}, (\Delta\mathscr{P})^2, (\Delta\mathscr{Q})^2, \Delta(\mathscr{Q}\mathscr{P}) \in \mathbb{R}$
  - positivity: $(\Delta\mathscr{P})^2, (\Delta\mathscr{Q})^2 \geq 0$
  - inequality: $(\Delta\mathscr{P})^2(\Delta\mathscr{Q})^2 - (\Delta(\mathscr{Q}\mathscr{P}))^2 \geq \frac{1}{4}\hbar^2$
- To interpret variables associated with $\mathbf{q}, \mathbf{p}$ as evolving in time $t = \langle \mathbf{t} \rangle$, fix 3 of the gauges: $\Delta(tp) = 0; \quad \Delta(qt) = 0; \quad (\Delta t)^2 = 0.$
- Inverting gauge-invariant functions recover dynamics:

$$p = \mathscr{P}; \quad q = \mathscr{Q} + \frac{\mathscr{P}}{M} t; \qquad (\Delta p)^2 = (\Delta\mathscr{P})^2;$$

$$(\Delta q)^2 = (\Delta\mathscr{Q})^2 + \frac{2\Delta(\mathscr{Q}\mathscr{P})}{M} t + \frac{(\Delta\mathscr{P})^2}{M} t^2; \quad \Delta(qp) = \Delta(\mathscr{Q}\mathscr{P}) + \frac{\Delta(\mathscr{P})^2}{M} t.$$

- These are the correct equations for a free quantum particle.

One final point

- In this example, once the constraints are imposed and 3 gauges are fixed, the dynamics may be recovered in two ways:

  1. As suggested here: by using $\langle \mathbf{t} \rangle$ to express the gauge dependence of variables generated by $\mathbf{q}, \mathbf{p}$.

  2. Using the remaining gauge flow generated on the constraint surface by $\langle \mathbf{C} \rangle = p_t + \frac{p^2}{2M} + \frac{(\Delta p)^2}{2M}$, i.e. taking Poisson bracket of $\langle \mathbf{C} \rangle$ with gauge dependent variables.

- In the absence of an obvious time variable 2nd method may be used to find a quantum parameter generating evolution.

Summary

- Goal: leading quantum corrections for constrained systems.
- Proceeded by:
  - treating $\Gamma \cong L(\mathscr{A} \to \mathbb{C})$ a Poisson phase-space
  - defining quantum constraints on Γ
  - reducing the system using a semi-classical expansion
  - enforcing reality, positivity, uncertainty on true observables
  - dynamics recovered as gauge correlation
- Advantages: straightforward procedure, solutions should be easy to perturb.
- Difficult to analyze stability of semi-classical approximation.
- Finally: quantum variables as clocks—uses in cosmology?

How can there be flows?

- Intuition: $\mathbf{C} \mid \psi\rangle = 0$ results in $\exp(\epsilon\mathbf{C}) \mid \psi\rangle = \mid \psi\rangle$ — no flow!
- Constraint functions fix the action of $\mathbf{C}$ on the maps from one side—the flows are generated by its action from the other side.
- Example 1: operators on a Hilbert space $dim(H) = N \quad (\mathscr{A} \cong U(N))$
  - suppose we have a vector $\mathbf{C} \mid \psi\rangle = 0$
  - any operator of the form $\rho = \mid \psi\rangle\langle\phi \mid$ where $\langle\phi \mid \psi\rangle = 1$, gives us a normalized linear map satisfying $Tr[(\mathbf{a}\mathbf{C})\rho] = 0, \forall \mathbf{a} \in \mathscr{A}$
  - unless we also have $\langle\phi \mid \mathbf{C} = 0$, there still is a gauge flow $\exp(-\epsilon\mathbf{C}) \mid \psi\rangle\langle\phi \mid \exp(\epsilon\mathbf{C}) = \mid \psi\rangle\langle\phi \mid \exp(\epsilon\mathbf{C})$
- Example 2: canonical pair as differential operators $\hat{x} = x, \;\; \hat{p} = i\hbar\frac{d}{dx}$
  - $\mathbf{C} = \hat{x}$ can be solved by any map of the form $\alpha_f(\hat{A}) = \delta_0[\hat{A}f(x)]$ with normalization $f(0) = 1$
  - $\mathbf{C}\alpha_f = 0$, but $\alpha_f\mathbf{C} = \alpha_{xf} \neq 0$ in general.
  - the corresponding flow is $\exp(-\epsilon\mathbf{C})\alpha_f \exp(\epsilon\mathbf{C}) = \alpha_{\exp(\epsilon x)f}$
- Note: functions derived from operators that commute with $\mathbf{C}$ are always gauge invariant.