GOWDY T^3 MODEL:

FOCK QUANTIZATION

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Loop Quantum Cosmology Workshop, IGC-Penn State, October 2008

MOTIVATION

- **Why are Gowdy models important?**
- **Can we handle the field theoretical Issues?**
- **Can we consistently quantize in any way?**
- **What can we learn from their quantization?**
- **Can we have control on the properties of the quantum theory? (uniqueness)**
- **Lessons from LQC in order to loop-quantize Gowdy?**

PLAN OF THE TALK

1. **Some History**

2. **Classical Preliminaries**

3. **Quantum Preliminaries**

4. **Quantum Theory I: Non-unitary evolution**

5. **Quantum Theory II: Unitary evolution**

6. **Uniqueness**

Work of many people, including F. Barbero, B. Berger, AC, J. Cortez, V. Husain, G. Mena-Marugan, C. Misner, M. Pierri, C. Torre, JM Velhinho, E. Villaseñor, D. Vergel, and more ...

1. SOME HISTORY

- **73' The Gowdy model is first quantized (Misner & Berger)**
- **80-90's Gowdy model reconsidered from the connection's perspective (Husain, Mena-Marugan)**
- **02' New systematic quantization (Pierri).**
- **02' Pierri's quantization shown to be non-unitary (AC, Cortez, Quevedo, Torre).**
- **05' New unitary quantization found (AC, Cortez, Mena-Marugan)**
- **06-07' Quantization is shown to be unique (AC, Cortez, Mena-Marugan, Velhinho)**
- **07-08' Results generalized to $S^2 \times S^1$ topology, including uniqueness (Barbero, Cortez, Mena-Marugan, Villaseñor)**

The Beginning

Rough idea: We want to go away from homogeneity but still have some control. Can we do it?

Yes! Answer: Polarized Gowdy T^3 models

They are the simplest (spatially closed) inhomogeneous cosmological models. They possess all the conceptual challenges of the homogeneous cosmologies, yet have an infinite number of degrees of freedom.

Classically they are nice: All exact solutions can be written explicitly.
They are thus natural testing ground for quantization procedures. They also contain (closed) Bianchi I as their homogeneous mode.

WHAT ARE GOWDY MODELS?

A polarized Gowdy spacetime, is a vacuum spacetime on $M = T^3 \times R$, with two commuting, hypersurface orthogonal Killing vector fields ∂_σ and ∂_δ. Thus, the metric only depends on one angle θ and time. One can write the metric in the following form:

$$\mathrm{d}s^2 = e^{\gamma}\, e^{-\phi/\sqrt{p}}(-\mathrm{d}t^2 + \mathrm{d}\theta^2) + e^{-\phi/\sqrt{p}}\, t^2 p^2\, \mathrm{d}\sigma^2 + e^{\phi/\sqrt{p}}\, \mathrm{d}\delta^2$$

where

$$\gamma = -\frac{\bar{Q}}{2\pi p} - \sum_{n=-\infty, n\neq 0}^{\infty} \frac{i}{2\pi n p} \oint \mathrm{d}\bar{\theta} e^{in(\theta-\bar{\theta})} P_\phi \phi' + \frac{1}{4\pi p} \oint \mathrm{d}\bar{\theta}\left[P_\phi \phi + P_\phi^2 + t^2(\phi')^2\right].$$

There is still a global constraint:

$$C_0 := \frac{1}{\sqrt{2\pi}} \oint \mathrm{d}\theta\, P_\phi \phi' = 0.$$

After a partial gauge fixing in the canonical theory, one arrives at a reduced description given by,

$$S_r = \int_{t_i}^{t_f} dt \left(\bar{P}\dot{\bar{Q}} + \oint d\theta \left[P_\phi \dot{\phi} - \mathcal{H}_r \right] \right), \qquad \mathcal{H}_r = \frac{1}{2t} \left[P_\phi^2 + t^2 (\phi')^2 \right]$$

which yields,

$$\ddot{\phi} + \frac{\dot{\phi}}{t} - \phi'' = 0.$$

which is the massless Klein Gordon equation on a fiducial metric:

$$g_{ab}^{\mathrm{B}} = -\nabla_a t \nabla_b t + \nabla_a \theta \nabla_b \theta + t^2 \, \nabla_a \sigma \nabla_b \sigma$$

Thus, for this classically reduced system, the problem of quantization of the geometry is reduced to that of quantizing a (symmetrical) massless scalar field on a fixed background, subject to a unique (quantum) constraint.

QUANTUM THEORY: PRELIMINARIES

A Fock Quantization:
Construct the 1-particle Hilbert space $\mathcal{H}$ out of the space $\mathcal{S}$ of classical solutions of th wave equation. A convenient way is to produce a complex structure J ($J : \mathcal{S} \to \mathcal{S}, J^2 = -\mathrm{Id}$). Then one can have a Hermitian inner product on $\mathcal{S}$:

$$\langle \cdot, \cdot \rangle := \Omega(\cdot, J\cdot) + i\,\Omega(\cdot, \cdot)$$

$\mathcal{H}$ is the Cauchy completion of $\mathcal{S}$ wrt $\langle \cdot, \cdot \rangle$.
The Hilbert space $\mathcal{F}_{\mathcal{H}}$ is then the symmetric Fock space.

When the spacetime is static we are in good shape (preferred choice of J). For a time dependent background, there is no canonical choice of J (or vacuum).

Can one have in this case physically motivate criteria for constructing such a unique theory?

QUANTUM THEORY I: NON-UNITARY EVOLUTION

Pierri's choice of complex structure is natural from the form of the solutions:

$$\varphi(t,\theta) = \sum_{n=-\infty}^{\infty} \left[A_n f_n(t,\theta) + A_n^* f_n^*(t,\theta)\right].$$

with $f_n(t,\theta) = e^{in\theta}\bar{f}_n(\theta) = e^{in\theta}\frac{H_0(|n|t)}{\sqrt{8}}$**. Then one can naturally define,**

$$\tilde{J}\big[\bar{f}_n(t)\big] = i\bar{f}_n(t), \qquad \tilde{J}\big[\bar{f}_n^*(t)\big] = -i\bar{f}_n^*(t).$$

The theory we are considering is constructed out of this choice of $\tilde{J}$, or alternatively, creation and anhilation operators.

Problem: The finite time evolution, that classically is a canonical transformation, is *not* unitarily implemented ! (the corresponding Bogoliubov coefficient is not square summable).

QUANTUM THEORY II: NEW PARAMETRIZATION

Let us define a new field parametrization:

$$\xi = \sqrt{t}\,\phi \qquad ; \qquad P_\xi := \frac{1}{\sqrt{t}}\left(P_\phi + \frac{\phi}{2}\right)$$

This change of variables is also a canonical transformation. What does it do? We have a new Hamiltonian:

$$\bar{H}_r = \frac{1}{2}\oint \mathrm{d}\theta \left[P_\xi^2 + (\xi')^2 + \frac{\xi}{4t^2}\right]$$

from which,

$$\ddot{\xi} - \xi'' + \frac{\xi}{4t^2} = 0.$$

which is the symmetric Klein Gordon field propagating on a static two-torus, but with a *time dependent potential* $V(t) = \frac{\xi^2}{8\,t^2}$**.**

We can now write the solutions as,

$$\zeta(t,\theta) = \sum_{n=-\infty}^{\infty} \left[A_n g_n(t,\theta) + A_n^* g_n^*(t,\theta)\right].$$

Where $g_n(t,\theta) := \sqrt{t} f_n(t,\theta)$**. by defining 'complex coordinates'**

$$b_n = \frac{|n|\xi_{(n)} + iP_\xi^{(n)}}{\sqrt{2|n|}}, \qquad b_{-n}^* = \frac{|n|\xi_{(n)} - iP_\xi^{(n)}}{\sqrt{2|n|}}.$$

we can rewrite,

$$\zeta(t,\theta) = \sum_{n=-\infty}^{\infty} \left[G_n(t,\theta) b_n(t_0) + G_n^*(t,\theta) b_n^*(t_0)\right].$$

$$G_n(t,\theta) = \sqrt{\frac{t}{8}} \left[c^*\left(x_{|n|}^0\right) H_0(x_{|n|}) - d^*\left(x_{|n|}^0\right) H_0^*(x_{|n|})\right] e^{in\theta}.$$

Here, $x_{|n|} = |n|t$ **and** $x_{|n|}^0 = |n|t_0$**.**

With this new decomposition comes a 'natural complex structure':

$$J\left[\bar{G}_n(t)\right] = i\bar{G}_n(t), \qquad J\left[\bar{G}^*_n(t)\right] = -i\bar{G}^*_n(t).$$

This 'innocent' change in field parametrization and complex structure has the striking property of 'curing' the unitarity problem.

The finite canonical transformation corresponding to time evolution is unitary.

$$\hat{b}^{(H)}_n(t) = \alpha_n(t,t_0)\hat{b}^{(H)}_n(t_0) + \beta_n(t,t_0)\hat{b}^{(H)\dagger}_{-n}(t_0),$$

where now the Bogoliubov coefficients $\beta_n(t,t_0)$ are square summable.

UNIQUENESS

The question here is: How unique is this quantization?
Or, in other words, can we select this quantization as preferred?
Answer: Yes!

Uniqueness Theorem (CCMV): If one requires:
i) Unitary evolution, and
ii) Invariance under the remaining constraint
There is a unique quantum representation!

Furthermore (CMV 07), among a large class of possible field reparametrizations, the only choice that satisfies i) and ii) is the one considered before. As a corollary, the Pierri field parametrization does not admit *any* quantum theory with these properties. Therefore, these conditions are sufficient to select unique quantum theory (like Poincare invariance in Minkowski or Diffeo invariance in LOST).

CONCLUSIONS

- **The Gowdy model is a suitable test ground for quantization**
- **One can consistently quantize the theory**
- **There exists a unitary quantum theory**
- **Asking for unitary time evolution and implementation of remaining constraint selects a unique theory**
- **Quantization is in logarithmic variables (Misner-like)**
- **This *is* the WDW theory with respect to which a loopy quantization has to be compared.**
- **Is there more?**

OUTLOOK

- **We have not dealt with the singularity**
- **We need a full description of the quantum geometry**
- **We need a full description of the semiclassical states**
- **A loopy quantization will probably not be in this kind of variables**