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HYBRID QUANTIZATION OF THE GOWDY COSMOLOGIES

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Motivation

- The loop quantization of homogeneous cosmological models has been studied recently. Besides, a satisfactory Fock quantization of inhomogeneous cosmologies has been achieved: the Gowdy model.

- The Gowdy T^3 model is a natural test bed to incorporate inhomogeneities in Loop Quantum Cosmology.

- The simplest possibility is a hybrid quantization.

- The initial singularity appears in the homogeneous solutions of the model (Bianchi I). How does the inclusion of inhomogeneities affect its quantum mechanical resolution?

- Does the loop quantization of the zero modes suffice to resolve the singularity? (Different from the “BKL” approach).

- Questions in mind are **internal time**, semiclassical behavior, validity of the Fock quantization, perturbative approaches...

Classical system

- We consider Gowdy T^3 cosmologies with linear polarization.
- The classical metric is (with $\theta, \sigma, \delta \in S^1$):

$$ds^2 = e^{\gamma[\phi]}\left(-dt^2 + d\theta^2\right) + t^2 e^{-\phi(t,\theta)} d\sigma^2 + e^{\phi(t,\theta)} d\delta^2 .$$

$$\phi(t,\theta) = \alpha + \beta \ln t + \sum_m [c_m J_0(mt) \sin(m\theta + \epsilon_m) + d_m N_0(mt) \sin(m\theta + \varepsilon_m)].$$

- Generically, $t = 0$ is a curvature singularity.
- Fixing the gauge, except for the zero modes of the θ-diffeos and scalar constraints, we get with a suitable field parametrization:

$$ds^2 = \frac{|p^1 p^2 p^3|}{4}\left[e^{\tilde{\gamma}[\xi, p^1]}\left(-\underline{N}^2 dt^2 + \frac{1}{(p^1)^2} d\theta^2\right) + \frac{e^{-\xi/\sqrt{|p^1|}}}{(p^2)^2} d\sigma^2 + \frac{e^{\xi/\sqrt{|p^1|}}}{(p^3)^2} d\delta^2\right].$$

$\underline{N}$ is homogeneous and ξ has no zero mode.

Choice of variables

- The zero modes can be viewed as the degrees of freedom of a Bianchi I model.

- In a diagonal gauge, the corresponding Ashtekar variables are

$$(\tilde{E}^{BI})_i^a = \frac{p_i}{4\pi^2}\delta_i^a, \quad (A^{BI})_a^i = \frac{c^i}{2\pi}\delta_a^i, \quad \{c^i, p_j\} = 8\pi G\gamma\delta_j^i.$$

- Expand the field and its momentum in Fourier modes, (ξ_m, P_ξ^m), and introduce the variables:

$$(a_m, a_m^*), \qquad a_m = \frac{|m|\xi_m + iK^2 P_\xi^m}{\sqrt{2|m|}K}, \quad K = \sqrt{\frac{4G}{\pi}}.$$

- The complex structure that is naturally associated with these variables determines a Fock space F^ξ.

- This is the **unique** Fock quantization with a unitary dynamics and a natural implementation of the remaining gauge group.

Remaining constraints

- The diffeomorphisms constraint generates S^1 translations.

$$C_\theta = \sum_{m>0} m\left(a_m^* a_m - a_{-m}^* a_{-m}\right).$$

It does not depend on the zero modes.

- Scalar constraint: Bianchi I plus the inhomogeneous Hamiltonian.

$$C_G := -\left(\frac{C_{BI}}{\gamma^2} + C_\xi\right),$$

$$C_{BI} = 2\,\frac{c^1 p_1 c^2 p_2 + c^1 p_1 c^3 p_3 + c^2 p_2 c^3 p_3}{\sqrt{|p_1 p_2 p_3|}}.$$

$$C_\xi = -\frac{4\pi^3 |p_1|}{\sqrt{|p_1 p_2 p_3|}}\left[\frac{(c^2 p_2 + c^3 p_3)^2}{16\pi^2\gamma^2 (p_1)^2}\sum |\xi_m|^2 + \sum\left\{\left(\frac{4G}{\pi}\right)^2 |P_\xi^m|^2 + m^2 |\xi_m|^2\right\}\right].$$

Bianchi I: representation

- We call $\{x^I\}=\{\theta,\sigma,\delta\}$. The Hilbert space H_{kin}^{BI} is the tensor product of $H_{kin}^{(I)}=L^2(\mathbb{R},d\mu_{Bohr}^I)$.

- We implement (a possibly modified version of) the proposal chosen by Chiou $\left(\bar{\mu}_I^{-1}\propto\widehat{\sqrt{|p_I|}}\right)$. We change to the v_I-basis.

- Using the standard methods of LQC:

$$\hat{C}_{BI}=\sum_{(I,J,K)}\hat{\Omega}_I\hat{\Omega}_J\left[\widehat{\frac{1}{\sqrt{|p_K|}}}\right],\qquad \left[\widehat{\frac{1}{\sqrt{|p_I|}}}\right]|v_I\rangle=\frac{1}{\sqrt{\gamma}\,l_p}b(v_I)|v_I\rangle,$$

$$\hat{\Omega}_I=a\widehat{\sqrt{|p_I|}}\left[\widehat{\sin(\bar{\mu}_Ic^I)}\,\widehat{sgn(p_I)}+\widehat{sgn(p_I)}\,\widehat{\sin(\bar{\mu}_Ic^I)}\right]\widehat{\sqrt{|p_I|}},\quad a=(8\sqrt{3}\pi\gamma l_p^2)^{-1/2}$$

- $\hat{C}^{BI}$ annihilates all the “zero volume” states: states in the basis $\{|v_1\rangle\otimes|v_2\rangle\otimes|v_3\rangle\}$ with any $v_I=0$. These states get **decoupled**.

In this sense, the singularity is resolved.

Bianchi I: Densitized constraint

- Restricting to the cylindrical functions and the kinematical Hilbert space $\bar{H}^{(I)}_{kin}$ without zero volume states, we densitize the constraint:

$$\hat{\tilde{C}}_{BI} = \left[\widehat{\frac{1}{\sqrt{|p_1 p_2 p_3|}}}\right]^{-1/2} \hat{C}_{BI} \left[\widehat{\frac{1}{\sqrt{|p_1 p_2 p_3|}}}\right]^{-1/2} = 2\left(\hat{\Lambda}_1\hat{\Lambda}_2 + \hat{\Lambda}_1\hat{\Lambda}_3 + \hat{\Lambda}_2\hat{\Lambda}_3\right),$$

$$\hat{\Lambda}_I |v_I\rangle = -\pi i \gamma l_p^2 \left(f_+(v_I)|v_I+2\rangle - f_-(v_I)|v_I-2\rangle\right),$$

$$f_\pm(v) = g(v\pm 2)\{sgn(v\pm 2) + sgn(v)\} g(v), \quad g(v) = \left| \left|1+\frac{1}{v}\right|^{1/3} - \left|1-\frac{1}{v}\right|^{1/3} \right|^{-1/2}$$

- $f_+(v_I)$ $\left(f_-(v_I)\right)$ vanishes in $[-2,0]$ $\left([0,2]\right)$. Then, $\hat{\Lambda}_I$ does not mix the semilattices $\mathscr{L}^2_{\pm\epsilon_I} := \{\pm(\epsilon_I + 2n),\ n \in \mathbb{N}\},\ \epsilon_I \in (0,2]$. The corresponding subspaces $\bar{H}^{(I)}_{\pm\epsilon_I}$ provide superselection sectors.

- In this sense, the constraint equation encodes a no-boundary.

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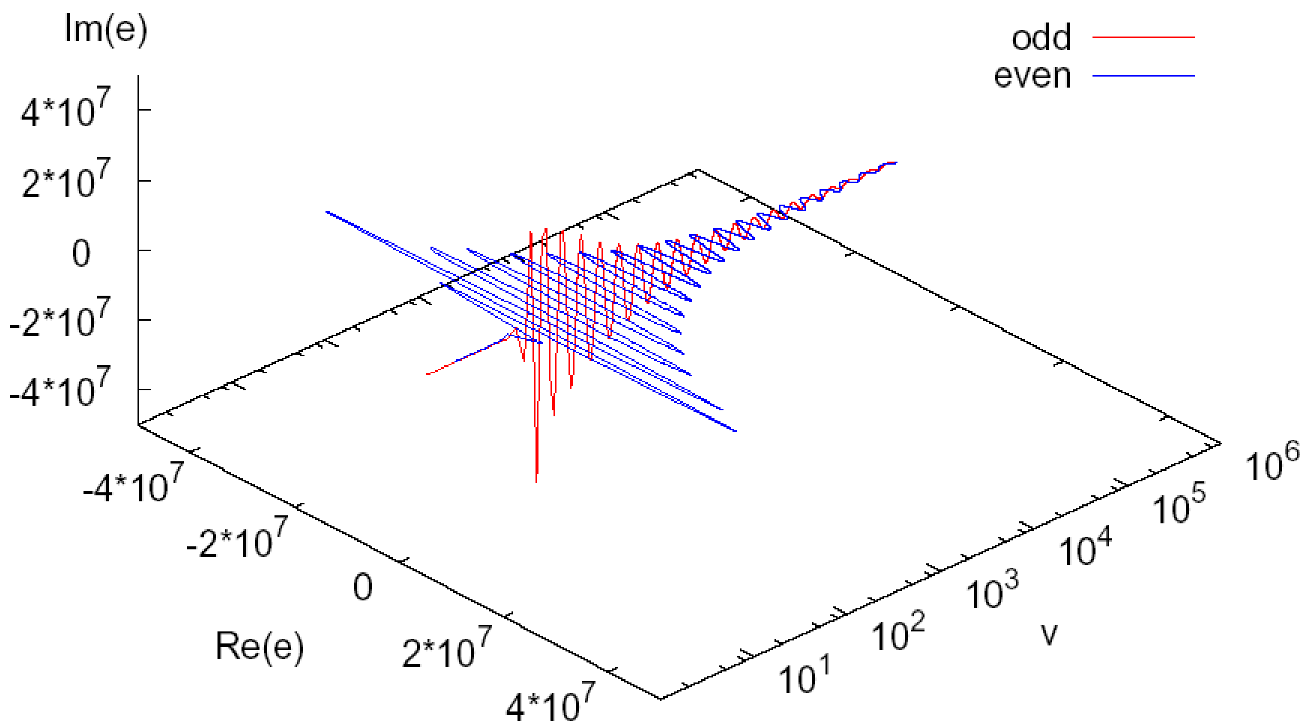
Spectrum and eigenfunctions of $\hat{\Lambda}_I$

- The WDW analog of $\hat{\Lambda}_I$ would be $12\pi i\gamma G v_I \frac{\partial}{\partial v_I}$.
- $\hat{\Lambda}_I$ (with domain the span of the v_I-states in the semilattice $\mathcal{L}^2_{\pm\epsilon_I}$) is essentially self-adjoint and has absolutely continuous spectrum.



$$1_{\pm\epsilon_I} = \int_{-\infty}^{\infty} d\lambda \left| e_\lambda^{\pm\epsilon_I} \right\rangle \left\langle e_\lambda^{\pm\epsilon_I} \right|, \qquad \hat{\Lambda}_I \left| e_\lambda^{\pm\epsilon_I} \right\rangle = \lambda \gamma l_p^2 \left| e_\lambda^{\pm\epsilon_I} \right\rangle.$$

Bianchi I: Physical states

- $$\hat{\tilde{C}}_{BI} = 2\left(\hat{\Lambda}_1\hat{\Lambda}_2 + \hat{\Lambda}_1\hat{\Lambda}_3 + \hat{\Lambda}_2\hat{\Lambda}_3\right).$$

  Since $\hat{\Lambda}_I$ are observables and we know their associated resolution of the identity, it is straightforward to solve the constraint.

- The same results can be obtained with group averaging. Physical states have the form

$$\psi(v_1, v_2, v_3) = \int_{-\infty}^{\infty} d\lambda_2 \int_{-\infty}^{\infty} d\lambda_3 \tilde{\psi}(\lambda_2, \lambda_3) e^{\epsilon_1}_{\lambda_1[\lambda]}(v_1) e^{\epsilon_2}_{\lambda_2}(v_2) e^{\epsilon_3}_{\lambda_3}(v_3)$$

with the Hilbert structure $\tilde{\psi} \in H^{BI} := L^2(\mathbb{R}^2, d\lambda_2 d\lambda_3 / |\lambda_2 + \lambda_3|)$ and

$$\lambda_1[\lambda] = -\lambda_2\lambda_3/(\lambda_2 + \lambda_3).$$

- A complete set of observables is given by $\hat{\Lambda}_2, \hat{\Lambda}_3, \hat{v}_2|_{v_1^0}, \hat{v}_3|_{v_1^0}$, for any given section v_1^0.

$$\left(\hat{v}_2|_{v_1^0}\right)\psi(v_1^0, v_2, v_3) = v_2\psi(v_1^0, v_2, v_3)$$

$$\Rightarrow \quad \left(\hat{v}_2|_{v_1^0}\right)\tilde{\psi}(\lambda_2, \lambda_3) = \int_{-\infty}^{\infty} d\tilde{\lambda}_2 \left\langle e^{\epsilon_2}_{\lambda_2} \middle| v_2 e^{\epsilon_2}_{\tilde{\lambda}_2} \right\rangle \tilde{\psi}(\tilde{\lambda}_2, \lambda_3).$$

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Hybrid Gowdy model

- The kinematical Hilbert space is $H_{kin}^{BI} \otimes F^{\xi}$.
- The inhomogeneous part $\hat{C}_{\xi}$ of the constraint annihilates also the zero volume states. Since these decouple, we can restrict ourselves to $\bar{H}_{kin} := \bar{H}_{kin}^{BI} \otimes F^{\xi}$.

We then arrive at the densitized constraint:

$$\hat{\tilde{C}}_G = -\frac{\hat{\tilde{C}}_{BI}}{\gamma^2} - \hat{\tilde{C}}_{\xi}, \qquad -\frac{\hat{\tilde{C}}_{\xi}}{l_p^2} = \frac{\left(\hat{\Lambda}_2 + \hat{\Lambda}_3\right)^2}{\gamma^2}\left[\widehat{\frac{1}{\sqrt{|p_1|}}}\right]^2 \hat{\mathrm{H}}_{Inter}^{\xi} + 32\pi^2 \widehat{|p_1|}\hat{\mathrm{H}}_0^{\xi},$$

$$\hat{\mathrm{H}}_{Inter}^{\xi} := \sum \frac{1}{2|m|}\left(2\hat{a}_m^{\dagger}\hat{a}_m + \hat{a}_m^{\dagger}\hat{a}_{-m}^{\dagger} + \hat{a}_m\hat{a}_{-m}\right), \quad \hat{\mathrm{H}}_0^{\xi} := \sum |m|\hat{a}_m^{\dagger}\hat{a}_m.$$

- We have represented the variables $c^I p_I$ by $\hat{\Lambda}_I$, like in Bianchi I. Then, $\hat{\Lambda}_2$ and $\hat{\Lambda}_3$ are **observables**, but $\hat{\Lambda}_1$ is not.

Densitized constraint

$$\hat{\tilde{C}}_G = -2\left(\hat{\Lambda}_1\hat{\Lambda}_2+\hat{\Lambda}_1\hat{\Lambda}_3+\hat{\Lambda}_2\hat{\Lambda}_3\right)+l_p^2\left\{\frac{\left(\hat{\Lambda}_2+\hat{\Lambda}_3\right)^2}{\gamma^2}\left[\widehat{\frac{1}{|p_1|}}\right]\hat{\mathrm{H}}^{\xi}_{Inter}+32\pi^2\widehat{|p_1|}\hat{\mathrm{H}}^{\xi}_0\right\},$$

$$\hat{\mathrm{H}}^{\xi}_{Inter}:=\sum\frac{1}{2|m|}\left(2\hat{a}^{\dagger}_m\hat{a}_m+\hat{a}^{\dagger}_m\hat{a}^{\dagger}_{-m}+\hat{a}_m\hat{a}_{-m}\right),\quad \hat{\mathrm{H}}^{\xi}_0:=\sum|m|\hat{a}^{\dagger}_m\hat{a}_m.$$

- If we view the constraint as an evolution equation, p_1 plays the role of **internal time**.
- Superselection: we restrict to $\bar{H}^{(1)}_{\epsilon_1}\otimes\bar{H}^{(2)}_{\epsilon_2}\otimes\bar{H}^{(3)}_{\epsilon_3}\otimes F^{\xi}$.
- We define $\hat{\tilde{C}}_G$ with domain the span of

$$\left\{|v_1\rangle\otimes|v_2\rangle\otimes|v_3\rangle\otimes|\{n_m\}\rangle:=|v_1,v_2,v_3,\{n_m\}\rangle;\, v_I\in\mathscr{L}^2_{\epsilon_I},\quad |\{n_m\}\rangle\in F^{\xi}\right\}.$$

The operator is well-defined and symmetric.

Eigenvalue equation: formal solutions

- The (complex) eigenvalue equation for $\hat{\tilde{C}}_G$ leads to

$$\left(\Psi\middle|\hat{\tilde{C}}_G\middle|v_1, v_2, v_3\{n_m\}\right\rangle = \rho\gamma^2 l_p^4 \left(\Psi\middle|v_1, v_2, v_3\{n_m\}\right\rangle, \quad \rho \in \mathbb{C}.$$

Substituting $\left(\Psi\right| = \sum_{v_1} \int_{\mathbb{R}^2} d\lambda_2 d\lambda_3 \langle v_1| \otimes \langle e^{\epsilon_2}_{\lambda_2}| \otimes \langle e^{\epsilon_3}_{\lambda_3}| \otimes \left(\Psi[v_1, \lambda_2, \lambda_3]\right|,$

we get
$$\left(\Psi[\epsilon_1 + 2M]\middle|\{n_m\}\right\rangle = \left(\Psi[\epsilon_1]\right| \sum_{\{r_i\}\cup\{s_j\}\in O(M)} \left[\Pi_{r_i} F(\epsilon_1 + 2r_i + 2)\right] \times P\left[\Pi_{s_j} \hat{\mathrm{H}}^{\xi}_{\rho}[\epsilon_1 + 2s_j]\right] \left|\{n_m\}\right\rangle,$$

$$F(v_1) := \frac{f_-(v_1)}{f_+(v_1)}, \qquad \hat{\mathrm{H}}^{\xi}_{\rho}[v_1] := \frac{i}{2\pi(\lambda_2+\lambda_3) f_+(v_1)}$$
$$\times \left[\rho + 2\lambda_2\lambda_3 - \frac{(\lambda_2+\lambda_3)^2}{\gamma} b^2(v_1) \hat{\mathrm{H}}^{\xi}_{Inter} - 2^6 3^{5/6} \pi^3 \gamma |v_1|^{2/3} \hat{\mathrm{H}}^{\xi}_0\right].$$

$O(M)$ is the set of paths from 0 to M with jumps of 1 or 2 steps.
$\{s_j\}$ are the points followed by a jump of 1 step.
P denotes path ordering.

Observables and physical states

- Solutions to the constraint correspond to $\rho=0$.

They are **determined** by the initial data $\left(\Psi[\epsilon_1]\right|$.

- If we identify solutions to the constraint with these initial data, observables are operators acting on them.

A complete set is provided by the observables for Bianchi I and, e.g., the operators representing

$$\left\{\left(\xi_m+\xi_{-m}, i\,\xi_m-i\,\xi_{-m}, P_\xi^m+P_\xi^{-m}, i\,P_\xi^m-i\,P_\xi^{-m}\right);\ \ m\in\mathbb{N}^+\right\}.$$

- With reality conditions we obtain (a Hilbert space equivalent to) $L^2(\mathbb{R}^2, d\lambda_2 d\lambda_3/|\lambda_2+\lambda_3|)\otimes F^\xi$.

- Imposing the S[1]-symmetry we get $L^2(\mathbb{R}^2, d\lambda_2 d\lambda_3/|\lambda_2+\lambda_3|)\otimes F^\xi_{phys}$.

F^ξ_{phys} is the subspace annihilated by $\hat{C}_\theta=\sum_{m>0} m(\hat{a}_m^\dagger\hat{a}_m-\hat{a}_{-m}^\dagger\hat{a}_{-m})$.

Conclusions

- By combining the loop quantization of Bianchi I (with compact sections) and the Fock quantization of the Gowdy model, we have constructed a hybrid quantization of these cosmologies in **vacuo**.

- We have obtained a well-defined constraint operator for the Gowdy model, found the solutions to the constraint and proceeded to determine the physical states and observables.

- The initial singularity is avoided (due to the loop quantization of the **zero modes**) and we get a no-boundary description.

- The physical Hilbert space is (equivalent to) that of the Fock quantization.

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