Effective equations

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What is Loop Quantum Cosmology?

Models full quantizations of gravity by loop methods, tries to analyze *physical effects.*

Exploiting several simplifying assumptions, *complete control* can sometimes be reached all the way to the physical Hilbert space.

Different types of assumptions, e.g. *symmetry* (homogeneity trivializes anomaly problem) or special *matter ingredients* (free massless scalar provides time, leads to solvability).

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Based on strong reductions, specific
"results cannot really be trusted." [J. Lewandowski: ILQG seminar]

To *model full theory reliably,* avoid ad-hoc choices and keep all required freedom in

- → equations: *matter content, refinement scheme,* and
- → initial values: *squeezed states, mixed states, potentially highly quantum states*

Tool: effective equations.

Holonomies

Choose coordinates, consider *co-moving region of coordinate volume* V_0. Expands/contracts by $V(t) = a(t)^3 V_0$.
Dynamics of $a(t)$ from Friedmann equation for isotropic components $\tilde{c}$ and $\tilde{p}$ of connection $A_a^i = \tilde{c}\delta_a^i$ and densitized triad $E_i^a = \tilde{p}\delta_i^a$. Poisson bracket V_0-independent for rescaled variables $c = V_0^{1/3}\tilde{c}$, $p = V_0^{2/3}\tilde{p}$.

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Basic ingredient to quantize curvature: “holonomies”

$$h = e^{i\ell_0\tilde{c}} = e^{i\ell_0 c/V_0^{1/3}} = e^{ic/\mathcal{N}^{1/3}} = e^{i\mu c}$$

along straight edges of coordinate length ℓ_0. If edges arranged homogeneously to form graph in region of size V_0, there are $\mathcal{N} = V_0/\ell_0^3$ vertices.

By construction: *$\mathcal{N}$ coordinate independent, h V_0-independent. Vertex density $\mathcal{N}/a^3V_0$ coordinate and V_0-independent.*

Lattice refinement

Constant co-moving ℓ_0: lattice is expanded by $a(t)$ to potentially macroscopic spacing. Avoid discreteness effects at large volume if new vertices created dynamically: $\mathcal{N}$ *grows* while ℓ_0 shrinks.

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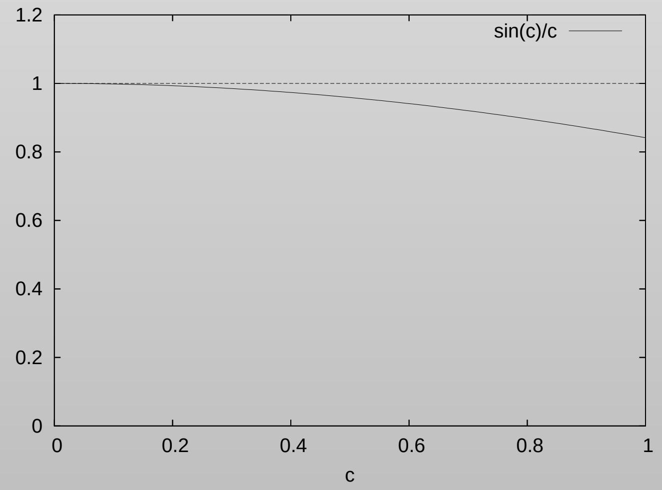
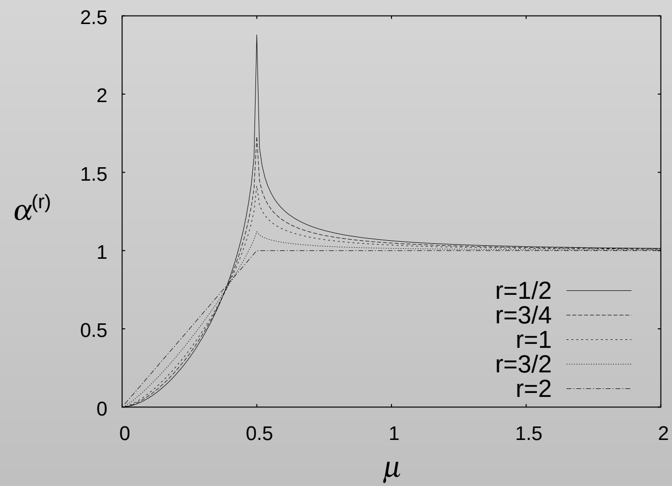
Also for *flux values* $\ell_0^2 a^2 = \ell_0^2 p / V_0^{2/3} = p/\mathcal{N}^{2/3}$ (inverse triad).

holonomy corrections significant for $\dot{a}/N > k_* := (\mathcal{N}/V_0)^{1/3}$

inverse triad corrections significant for $a < a_* := (\mathcal{N}/V_0)^{1/3} \ell_{\rm P}$

Holonomy corrections decreasing for larger $\mathcal{N}$, inverse triad corrections increasing: *change balance of corrections just by refining.*





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Power-law: $\mathcal{N} = \mathcal{N}_0 a^{-6x}$, $0 < x < -1/2$ (limiting case $\mathcal{N} \propto V$).

$\mathcal{N}_0$ may depend non-trivially on V_0 and coordinates: coordinate dependent scale factor used. If this is ignored, $\mathcal{N} \propto V_0 a^3$ and thus $x = -1/2$ ($\bar{\mu}$) is the only possibility.

Uniqueness “proof” flawed: possibility of non-trivial coordinate dependence of $\mathcal{N}_0$ is excluded by hand.

Dynamics

For given refinement, Hamiltonian constraint to be quantized:

$$-\mathcal{N}^{2/3}\sin^2\left(\frac{c}{\mathcal{N}^{1/3}}\right)\sqrt{|p|}+\frac{8\pi G}{3}\left(\frac{p_\phi^2}{2|p|^{2/3}}+|p|^{2/3}W(\phi)\right)=0$$

Quantization can be analyzed by *effective equations* which in general *do not agree* with the simple modified Hamiltonian containing $\mathcal{N}^{2/3}\sin^2(c/\mathcal{N}^{1/3})$ instead of the classical c^2.

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Ineffective equation: simply use classical modification. In general amounts only to *tree level, no quantum corrections.*

It is the (almost) complete effective equation for the *free theory* $W(\phi)=0$, which represents a *solvable, harmonic system:* no loop corrections, no quantum back-reaction.

Does not generalize to $W(\phi)\neq 0$, just as the effective potential of the harmonic oscillator (no quantum corrections) does not generalize to anharmonic systems.

Harmonic cosmology

Instead: $\langle \hat{C} \rangle = C_Q(\langle \widehat{e^{ic}} \rangle, \langle \hat{p} \rangle, \Delta e^{ic}, \Delta p, C_{e^{ic}p}, \ldots)$ as *effective constraint;* see A. Tsobanjan's talk.

Complicated system, but can be deparameterized first to realize solvability in free case.

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Specific factor ordering, ignoring factors: $p_\phi = \pm|\mathrm{Im}J|$ where $J = p^{1-x}\exp(ip^x c)$. With $V = p^{1-x}$: linear algebra

$$[\hat{V}, \hat{J}] = \hbar\hat{J} \quad , \quad [\hat{V}, \hat{J}^\dagger] = -\hbar\hat{J}^\dagger \quad , \quad [\hat{J}, \hat{J}^\dagger] = -2\hbar\hat{V} - \hbar^2$$

With "linear" Hamiltonian, *solvability* is implied:

$$\frac{\mathrm{d}\langle\hat{V}\rangle}{\mathrm{d}\phi} = \frac{\langle[\hat{V},\hat{H}]\rangle}{i\hbar} = -\tfrac{1}{2}(\langle\hat{J}\rangle + \langle\hat{J}^\dagger\rangle)$$

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Positivity

$p_\phi = \pm|\mathrm{Im}J|$ linear if $\mathrm{Im}J$ is positive or negative definite.

Positive frequency: take $p_\phi = \mathrm{Im}J =: H$ and start with initial moments of a state supported only on the positive part of $\mathrm{spec}H$. Since H preserved during ϕ-evolution, support on $\mathrm{spec}H$ remains positive.

Due to linearity, there is *no quantum back-reaction* in evolution; positivity only provides restrictions of (high order) initial moments.

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Alternatively, take $p_\phi = -H$ and a state supported only on the negative part of $\mathrm{spec}H$.

Superpositions of $\pm H$ states allowed, but moments of whole superposition not as interesting as moments of individual contributions.

States through a bounce

Equations of motion for expectation values decouple:

$$\frac{\mathrm{d}\langle\hat{V}\rangle}{\mathrm{d}\phi} = \frac{\langle[\hat{V},\hat{H}]\rangle}{i\hbar} = -\tfrac{1}{2}(\langle\hat{J}\rangle + \langle\hat{J}^\dagger\rangle)$$

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with general solution

$$\langle\hat{V}\rangle(\phi) = \tfrac{1}{2}(Ae^{-\phi} + Be^{\phi}) - \tfrac{1}{2}\hbar$$

$$\langle\hat{J}\rangle(\phi) = \tfrac{1}{2}(Ae^{-\phi} - Be^{\phi}) + iH$$

“Bounce” since $|\langle\hat{V}\rangle| \to \infty$ for $\phi \to \pm\infty$, but *could enter deep quantum regime* if $AB < 0$; solvable model would break down.

Reality condition

Classical: $J\bar{J} = V^2$. Quantum: $\hat{J}\hat{J}^\dagger = \hat{V}^2$. Related to *physical inner product:* $\exp(iP)$ becomes unitary operator.

Expectation values: $|\langle\hat{J}\rangle|^2 - (\langle\hat{V}\rangle + \frac{1}{2}\hbar)^2 = G^{VV} - G^{J\bar{J}} + \frac{1}{4}\hbar^2$.

Implies $AB = H^2 + O(\hbar)$, *bouncing* solution ($e^{2\delta} = B/A$)

$$\langle\hat{V}\rangle(\phi) = H\cosh(\phi - \delta) \quad , \quad \langle\hat{J}\rangle(\phi) = -H(\sinh(\phi - \delta) - i)$$

$$\rho = \frac{p_\phi^2}{2p^3} = \frac{H^2}{2V^{3/(1-x)}}$$

$\longrightarrow$ energy density bounded above independently of p_ϕ for $x = -1/2$, but not for $x \neq -1/2$.

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Uncertainties (e.g. $G^{VV} = \langle\hat{V}^2\rangle - \langle\hat{V}\rangle^2$):

$$\dot{G}^{VV} = -G^{VJ} - G^{V\bar{J}} \quad , \quad \dot{G}^{JJ} = -2G^{VJ} \quad , \quad \dot{G}^{\bar{J}\bar{J}} = -2G^{V\bar{J}}$$

$$\dot{G}^{VJ} = -\frac{1}{2}G^{JJ} - \frac{1}{2}G^{J\bar{J}} - G^{VV} \, , \, \dot{G}^{V\bar{J}} = -\frac{1}{2}G^{\bar{J}\bar{J}} - \frac{1}{2}G^{J\bar{J}} - G^{VV}$$

$$\dot{G}^{J\bar{J}} = -G^{VJ} - G^{V\bar{J}}$$

$\longrightarrow$ Reality preserved: $\dot{G}^{VV} - \dot{G}^{J\bar{J}} = 0$.

$\longrightarrow$ Near saturation, $(\Delta V)^2 = G^{VV} \approx \hbar H\cosh(2(\phi - \delta_2))$.

In general, $\delta_2 \neq \delta$: asymmetric fluctuations.

On the momentous role of moments

Fluctuations and higher moments dynamical, in general back-react on expectation values. Affect results of effective equations and must be analyzed properly.

Example: Compute $\langle \hat{C} \rangle$ in Gaussian state *assumed to be realized at all times.* [J. Willis; V. Taveras; Y. Ding, Y. Ma, J. Yang]
Gives ineffective constraint with $\sin^2(c/\mathcal{N}^{1/3})$ modification, plus corrections of an order given by the (possibly phase-space dependent) variance of the state.
Assume: $\Delta V \propto V^r$ for some parameter r. *Recollapse* for $r = 0$.
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In this scheme *only weak control over state,* but dynamics of ΔV must be determined in quantum theory. Equation of motion for moments: $(\Delta V)/V \approx \mathrm{const}$, thus $r=1$ and no recollapse (or other large-volume effect) happens.

Lest the universe forgets

Dynamical coherent states:

$$D := \left| \lim_{\phi\to-\infty} \frac{G^{VV}}{\langle\hat{V}\rangle^2} - \lim_{\phi\to\infty} \frac{G^{VV}}{\langle\hat{V}\rangle^2} \right|$$

$$= 4\frac{H}{A}\sqrt{\left(1-\frac{H^2}{A^2}+\frac{1}{4}\frac{\hbar^2}{A^2}\right)\frac{(\Delta H)^2}{A^2}-\frac{1}{4}\frac{\hbar^2}{A^2}+\left(\frac{H^2}{A^2}-1\right)\frac{(\Delta H)^4}{A^4}}$$

of order fluctuations *squared.*

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of order fluctuations *squared.* Compare with [Corichi, Singh]

$$D \leq 8\beta\epsilon \sim 8\Delta P/P$$

right hand side *linear in relative fluctuations.*

Example: $\beta\epsilon_+ \approx \alpha_2(\Delta V/V)_+ \approx \sqrt{\alpha_2}10^{-57}$, thus $D \leq \sqrt{\alpha_2}10^{-57}$.

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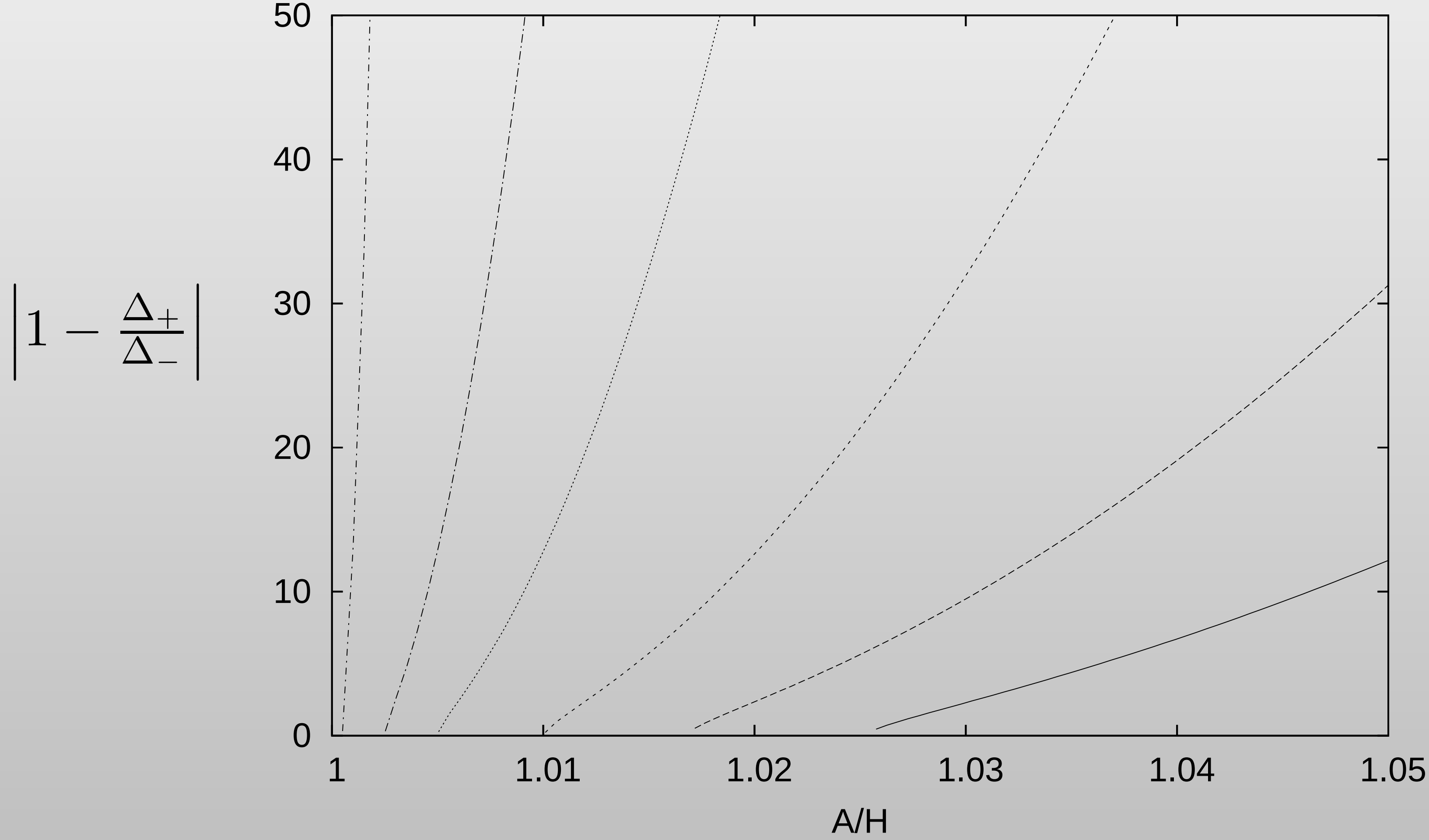
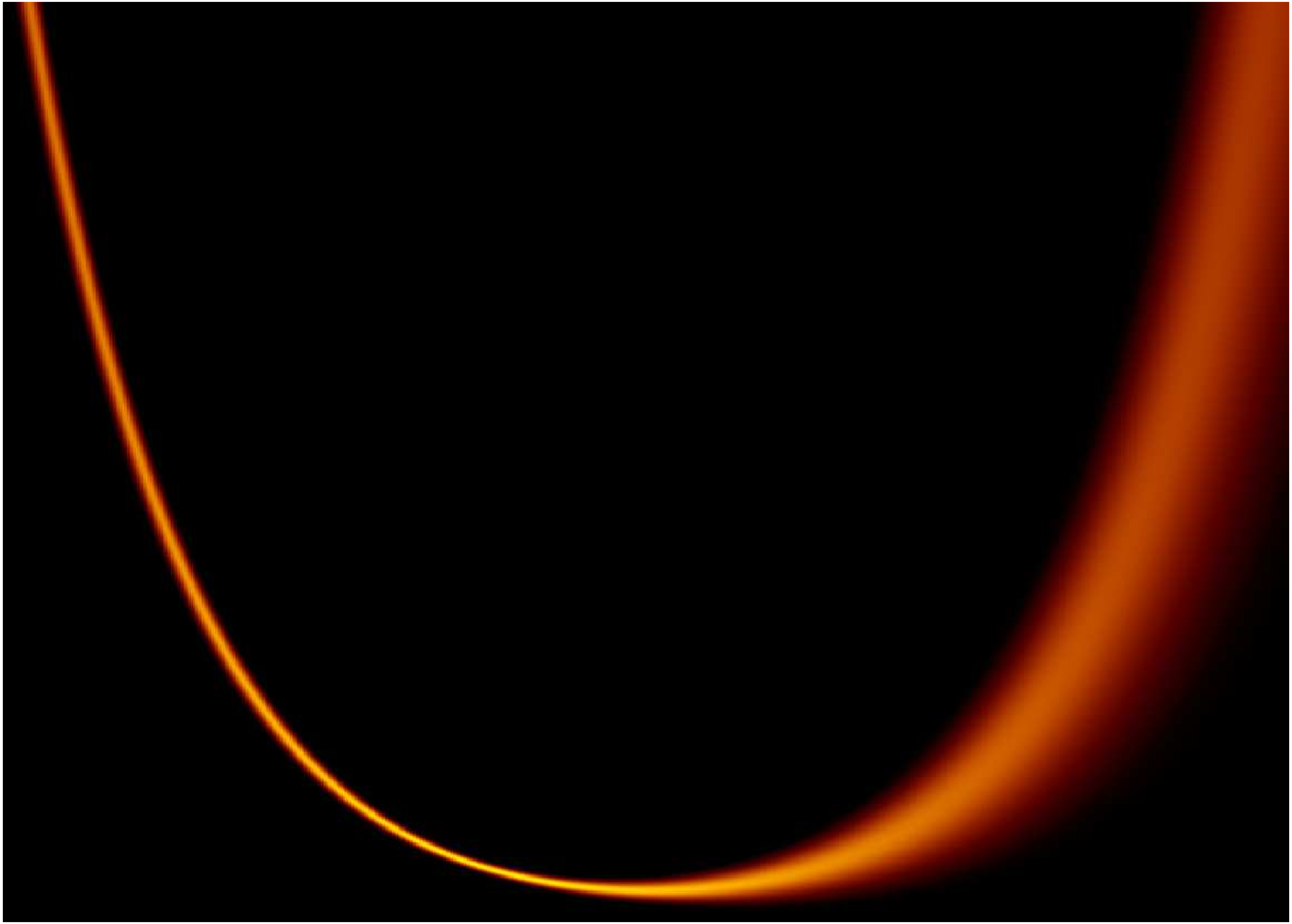
$(\Delta V/V)_-^2 \leq \sqrt{\alpha_2}10^{-57} + (\Delta V/V)_+^2 \approx \sqrt{\alpha_2}10^{-57}$, thus
$(\Delta V/V)_- \approx \alpha_2^{1/4}10^{-28} \approx \alpha_2^{3/4}10^{28}(\Delta V/V)_+$ ["(almost) total recall"]

Cosmic forgetfulness

Asymmetry of fluctuations before and after the bounce extremely sensitive to initial values (A, H, ΔH).

For different values of $H = \langle \hat{H} \rangle$, steepness increasing:



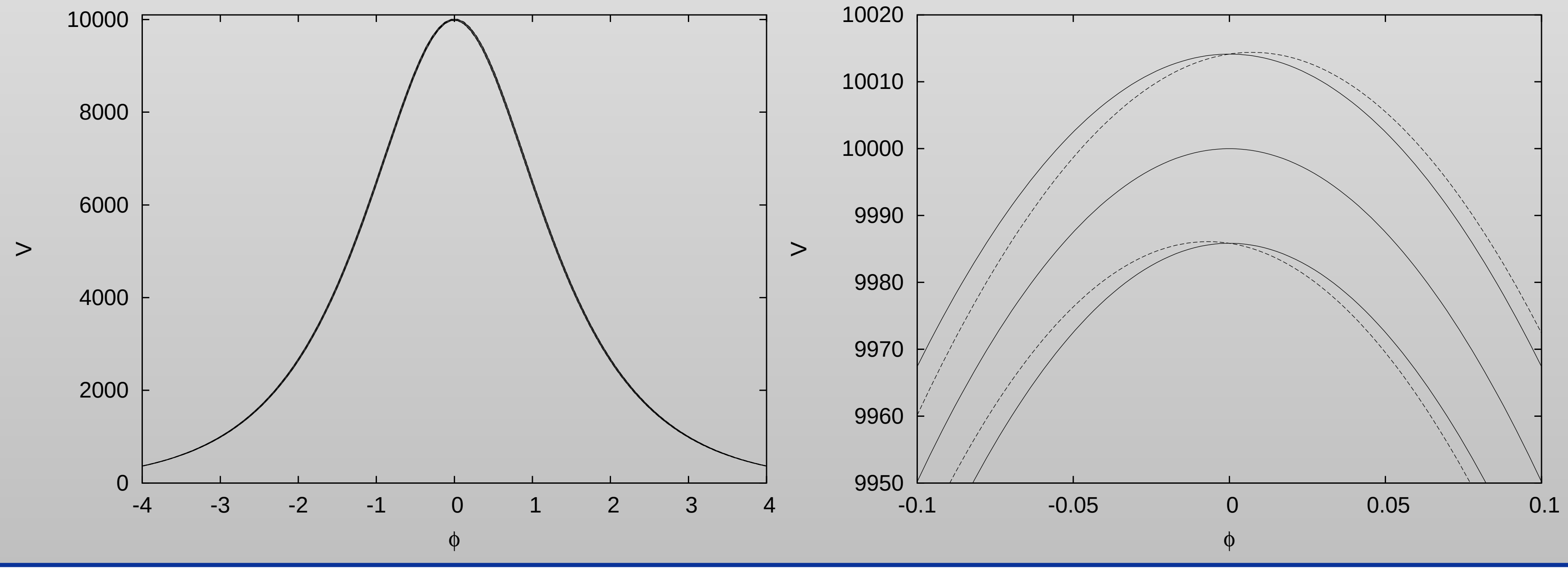


Perturbing harmonic cosmology

Deviations from free model imply quantum back reaction.

$\longrightarrow$ Cosmological constant Λ: Hamiltonian still preserved, no positivity problem, expand Hamiltonian around free model.

If $\Lambda < 0$, quantum back-reaction not relevant at large volume: semiclassicality preserved over long times; similar if positive spatial curvature. *Cyclic models,* but fluctuations increase over several cycles: $\mathrm{d}G^{VV}/\mathrm{d}\phi \sim VG^{VP}$ [MB, R. Tavakol]



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For $\Lambda > 0$, some moments diverge where $\langle \hat{V} \rangle$ diverges: *quantum back-reaction essential,* $\sin^2 c$ not sufficient. "Recollapses"?

$\longrightarrow$ Big rip singularities: ineffective equation singular. [T. Cailleteau, A. Cardoso, K. Vandersloot, D. Wands] Asymptotes to $V = \mathrm{const}$, $\Delta\phi$ *large.*

$\longrightarrow$ $W(\phi) \neq 0$: time-dependent Hamiltonian, not preserved.

Quantum back-reaction especially of correlations affects bounce condition. (Possibly other corrections to ensure positivity.)

Kinetic-dominated quantum Friedmann eq.

$[x = -1/2]$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho\left(1-\frac{\rho_Q}{\rho_{\rm crit}}\right)\right.$$
$$\left.\pm\frac{1}{2}\sqrt{1-\frac{\rho_Q}{\rho_{\rm crit}}}\eta(\rho-P)+\frac{(\rho-P)^2}{(\rho+P)^2}\eta^2\right)$$

where P is pressure and η parameterizes quantum correlations,

$$\rho_Q := \rho+\epsilon_0\rho_{\rm crit}+(\rho-P)\sum_{k=0}^{\infty}\epsilon_{k+1}(\rho-P)^k/(\rho+P)^k$$

with fluctuation parameters ϵ_k; $\rho_{\rm crit} = 3/8\pi G\mu^2$ with scale μ.

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Simple behavior if $\rho = P$ (free, massless scalar):
bounSingh. [P. Singh, PRD 73 (2006) 063508]

Also if $\eta = 0$ when $\rho_Q = \rho_{\rm crit}$ (no correlations).

For $x \neq -1/2$, bounce density depends on initial values, but also on x and $\mathcal{N}_0$ of refinement scheme (even in free case).

Lewandowski doctrine

Specific numbers (e.g. bounce density, growth of fluctuations over many cycles based on special initial state) are *parameter dependent* (refinement, class of states).

But *general parameter-freedom,* e.g. x combined with $\mathcal{N}_0$ can be retained and is already constrained by *phenomenological*

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Effective equations allow an analysis of the behavior of *generic states: squeezed, mixed, possibly highly quantum near the big bang.*

Rather than specific but too special values, “negative” statements can be derived in the form of *limitations.* Reliable if they even occur in highly simplified (possibly solvable) models.