Spinors in Coordinates: How Ogievetsky and Polubarinov Avoid a Tetrad

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Spinors and Arbitrary Coordinates?

- Use of arbitrary coordinates traditional in GR.
- Einstein, 1916: can't adapt coordinates to simplify laws in GR, so allow arbitrary coordinates.
- Are there spinors (under Lorentz transformations) that can be written in arbitrary coordinates?
- Most say no, but in fact there are (Ogievetskiĭ and Polubarinov, 1965).
- Geometric Objects: for each space-time point and local coordinate system, components and transformation law (Nijenhuis, 1952; Kucharzewski and Kuczma, 1964; Trautman, 1965).

- E.g., $v^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} v^{\nu}$.
- Tensors, densities are linear homogeneous: $v' \sim v$.
- Connections linear inhomogeneous: $\Gamma' \sim \Gamma + O(0)$.
- Symmetric square root of metric $r_{\mu\nu}$: $g_{\mu\nu} = r_{\mu\alpha}\eta^{\alpha\beta}r_{\beta\nu}$, $\eta^{\alpha\beta} = diag(-1, 1, 1, 1)$ a matrix, not a metric.
- $r_{\mu\nu}$ is (almost) nonlinear g.o.: $r' \sim$ even series in r.
- $r' \sim r$ linear for 15-parameter conformal group (stability group).
- Jets like $\langle g_{\mu\nu}, g_{\mu\nu,\alpha} \rangle$. Modern view: natural bundles (Nijenhuis, 1972; Fatibene and Francaviglia, 2003).
- Transformation rule gives Lie derivative (Szybiak, 1966), (when meaningful) covariant derivative (Szybiak, 1963).
- Transformations near 1 suffice (Szybiak, 1963; Szybiak, 1966).

- Lie, covariant derivatives of nonlinear g.o. χ: only *pair* $\langle \chi, \mathcal{L}_\xi \chi \rangle$ is a g.o.; likewise $\langle \chi, \nabla \chi \rangle$ (Yano, 1957; Szybiak, 1966; Szybiak, 1963).
- Do spinors fit in somehow?
- James L. Anderson discusses spinors, absolute objects and general covariance (Anderson, 1967), but doesn't use g.o.!
- Majority view: no, so use orthonormal tetrad e^μ_A; coordinate μ, Lorentz A; spinor as coordinate scalar, Lorentz spinor.
- Since Cartan, 'everyone knows' spinors not representation of general coordinate transformations; need for Lorentz group transformations, orthonormal tetrad (Weinberg, 1972; Deser and Isham, 1976; van Nieuwenhuizen, 1981; Lawson and Michelsohn, 1989; Fatibene and Francaviglia, 2003).

- But notice that tetrad leg, given coordinate and gauge freedom, can be $(1, 0, 0, 0)$ in any neighborhood. Thus an absolute object in GR + electron field?! (Pitts, 2006) *C.f.* (Anderson, 1967; Thorne et al., 1973; Lee et al., 1974).
- James L. Anderson, Thorne-Lee-Lightman-Ni emphasize removing *irrelevant* fields in testing a theory for general covariance.
- $\frac{6}{16}$ of tetrad is irrelevant. Let's *eliminate* it.
- 'Everyone' refuted in 1960s by Ogievetsky and Polubarinov (OP) (Ogievetskiĭ and Polubarinov, 1965; Huggins, 1962; Borisov and Ogievetskii, 1974; Bilyalov, 2002) with symmetric square root of metric $r_{\mu\nu}$ formalism.

- Can symmetrize tetrad as gauge-fixing of tetrad's local Lorentz freedom (DeWitt and DeWitt, 1952; Isham et al., 1971; Cho and Freund, 1975; Boulware et al., 1979).
- OP: $r_{\mu\nu}$ *itself* represents arbitrary infinitesimal coordinate transformations, nonlinearly, with only coordinate indices.
- Spinors in coordinates not impossible, just hard: spinors in nonlinear representation of coordinate transformations, linear for global Lorentz. (Gates et al., 1983, p. 234)
- Can add local Lorentz group for *convenience*: "by enlarging the gauge group, we obtain linear spinor representations. The nonlinear spinor representations of the general coordinate group reappear only if we fix a gauge for the local Lorentz transformations." (Gates et al., 1983)

- OP mathematically refined, made less perturbative (Bilyalov, 2002; Bilyalov, 1992; Bourguignon and Gauduchon, 1992).
- As if choosing local Lorentz freedom to symmetrize tetrad $e^{\mu A} = e^{\alpha M}$, inferring spinor coordinate transformation from spinor Lorentz transformation.
- But $r_{\mu\nu}$ independent of tetrad, even exists when tetrad is topologically obstructed. $r_{\mu\nu}$ is (almost?) a g.o., analytic function of $g_{\mu\nu}$.
- Hairy ball theorem on sphere (Spivak, 1979), Stiefel-Whitney class restrictions (DeWitt et al., 1979) can exclude tetrad.
- OP: spinor ψ *with* $r_{\mu\nu}$ forms representation of general infinitesimal coordinate transformations.
- Numerical Dirac γ matrices: independent of $g_{\mu\nu}$, coordinates.

- Oddly, no-go theorem for spinors and OP refutation coexist over 40 years. No published criticisms of OP.
- Hope that whoever is wrong, will notice.
- OP: Cartan's no-go theorem lacks imagination. Not ψ, but $\langle r_{\mu\nu}, \psi \rangle$; ψ transformation depends on $r_{\mu\nu}$.
- So one needs only 10 components of $r_{\mu\nu}$, not 16 of e^{μ}_{A}.
- GR + spinor fits natural (coordinate) bundle, no need for gauge-natural bundle.
- $\langle r_{\mu\nu}, \psi \rangle$ equivalent as g.o. to $\langle g_{\mu\nu}, \psi \rangle$.
- Removing irrelevant fields from spinor-tetrad formalism removes absolute object from GR + spinor.
- $\langle r_{\mu\nu}, \pm\psi \rangle$ to address spinor 2-valuedness; geometric pseudo-object (Siwek, 1965).

- If spinors ψ & χ, $\langle r_{\mu\nu}, \pm\psi, \pm\chi \rangle$ with same signs.
- Are just any coordinates allowed? Does it matter?
- Binomial series:

$$r^{\mu\nu} = \sum_{k=0}^{\infty} \frac{\frac{1}{2}!}{(\frac{1}{2}-k)!k!} [(g^{\mu\bullet} - \eta^{\mu\bullet})\eta_{\bullet\bullet} \ldots (g^{\bullet\nu} - \eta^{\bullet\nu})]^{k \ factors}$$

$$= \eta^{\mu\nu} + \frac{1}{2}(g^{\mu\nu} - \eta^{\mu\nu}) - \frac{1}{8}(g^{\mu\alpha} - \eta^{\mu\alpha})\eta_{\alpha\beta}(g^{\beta\nu} - \eta^{\beta\nu}) + \ldots .$$

- Convergence for coordinates not too far from Cartesian.
- Bilyalov's eigenvector formalism more general (Bilyalov, 2002), but not quite fully: put 'time' first.
- Need for arbitrary coordinates? Entrenched habit, not a fact.
- Equivalence principle only needs some bends and wiggles.

Differentiating OP Spinors as Nonlinear Geometric Objects

- Unlike other spinors, OP spinors have classical Lie derivative: $\langle r_{\mu\nu}, \psi, \mathcal{L}_\xi r_{\mu\nu}, \mathcal{L}_\xi \psi \rangle$ is a g.o.; $[\mathcal{L}_\xi, \mathcal{L}_\phi]\psi = \mathcal{L}_{[\xi,\phi]}\psi$.
- $\langle r_{\mu\nu}, \psi \rangle$ equivalent g.o. to $\langle g_{\mu\nu}, \psi \rangle$. $\nabla g_{\mu\nu} = 0 \leftrightarrow \nabla r_{\mu\nu} = 0$.
- Only conformal metric density needed: $\langle \hat{g}_{\mu\nu}, \psi \rangle$ with $|\hat{g}_{\mu\nu}| = -1$, $g_{\mu\nu} = \hat{g}_{\mu\nu}\sqrt{-g}^{\,\frac{1}{2}}$.
- Transformation rules for $\mathcal{L}_\xi \chi$, $\nabla \chi$, for χ a nonlinear g.o. (Szybiak, 1963; Szybiak, 1966):
- $(\mathcal{L}_\xi \psi)' \sim \left(\frac{\partial \psi'}{\partial \psi} \mathcal{L}_\xi \psi + \frac{\partial \psi'}{\partial \hat{g}_{\mu\nu}} \mathcal{L}_\xi \hat{g}_{\mu\nu} \right)$: Lie derivative with respect to conformal Killing vector is *nicer*, but exists in general.
- $(\nabla \psi)' \sim \left(\frac{\partial \psi'}{\partial \psi} \nabla \psi + \frac{\partial \psi'}{\partial \hat{g}_{\mu\nu}} \nabla \hat{g}_{\mu\nu} \right) \sim \left(\nabla \psi + \frac{\partial \psi'}{\partial \hat{g}_{\mu\nu}} \nabla \hat{g}_{\mu\nu} \right)$.
- Using $\nabla \hat{g}_{\mu\nu} = 0$, $\nabla \psi$ is spinor-covector, $\langle \hat{g}_{\mu\nu}, \nabla \psi \rangle$ is g.o.

- $\mathcal{L}_\xi$, ∇ need only infinitesimal transformations.
- Amount of coordinate conventionality is conventional.
- Einstein, 1916: coordinates directly yielding lengths impossible in GR, "and there seems to be no other way which would allow us to adapt systems of co-ordinates to the four-dimensional universe so that we might expect from their application a particularly **simple** formulation of the laws of nature. So there is nothing for it but to regard all imaginable systems of co-ordinates, on principle, as equally suitable for the description of nature." (emphasis added) (Einstein, 1923, p. 117)
- Arbitrary coordinates a convention, reasonable until troublesome.

- 1960s OP spinors: either $r_{\mu\nu}$ nonperturbative, or using only coordinates for which $r_{\mu\nu}$ series converges would *deflate* GR + spinor by 6 fields *vs.* tetrad.
- Einstein's simplicity criterion might now suggest coordinate restriction for OP spinors.
- If OP excludes arbitrary coordinates, no big loss.
- Precedent: metric inequalities enforcing one time, three space coordinates (Hilbert, 1917; Pauli, 1921; Møller, 1952).
- Particle physics: parity inversion P, T time reversal metric-dependent.
- Examples: allowed transformations depend on metric.
- Brandt groupoid (Hahn, 1978; Renault, 1980), not group of coordinate transformations: can't multiply all elements.

- Bilyalov (Bilyalov, 2002) accommodates arbitrary set of coordinates, but not in *arbitrary* order.
- Often cannot transform $\langle t, x, y, z \rangle \rightarrow \langle x, t, y, z \rangle$; avoid wrong 'time.'
- All the coordinates that one needs, and most that one wants.
- Restriction reflects Lorentz group in particle physics.
- (Misner) Q: Schwarzschild radius—same coordinate changes sign?
- A. Maybe use illicit coordinates for geometry, not spinors.
- Giving up a bit of coordinate freedom, if necessary, is justified by accommodating spinors without extra fields, extra gauge group and loss of Lie derivative.

Conclusions and Questions

- GR + spinors avoids Anderson-Thorne-Lee-Lightman absolute object only by OP-Bilyalov symmetric $r_{\mu\nu}$ (Pitts, 2006).
- $r_{\mu\nu}$ restricts *order* of coordinates, which one is time.
- Arbitrary coordinates permitted? Why? Conventional choice.
- Simplicity of fewer fields $\rightarrow$ OP with coordinate restrictions. Simplicity of linear transformation law $\rightarrow$ tetrad, no coordinate restrictions, extra local $O(3,1)$, lose Lie derivative.
- OP spinors in modern natural bundle context? Help welcome.
- OP spinors and topology—more general than tetrad formalism (other signatures or dimensions (Choquet-Bruhat et al., 1996))? Help welcome.

Appendix: Groupoids A (Brandt) groupoid (Hahn, 1978; Renault, 1980) is a set G endowed with product map $(x, y) \rightarrow xy : G^2 \rightarrow G$, where G^2 (a subset of $G \times G$) is the set of composable ordered pairs, and an inverse map $x \rightarrow x^{-1} : G \rightarrow G$ such that:

1. $(x^{-1})^{-1} = x$,
2. if (x, y) and (y, z) are both elements of G^2, then (xy, z) and (x, yz) are also elements of G^2 and $(xy)z = x(yz)$,
3. $(x^{-1}, x) \in G^2$, and if $(x, y) \in G^2$, then $x^{-1}(xy) = y$,
4. $(x, x^{-1}) \in G^2$, and if $(z, x) \in G^2$, then $(zx)x^{-1} = z$.

- Groupoid multiplication is associative when defined.
- Each element has a 2-sided inverse.
- Many little identity elements, such as xx^{-1}.

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