Two timescale expansions of the Einstein equations

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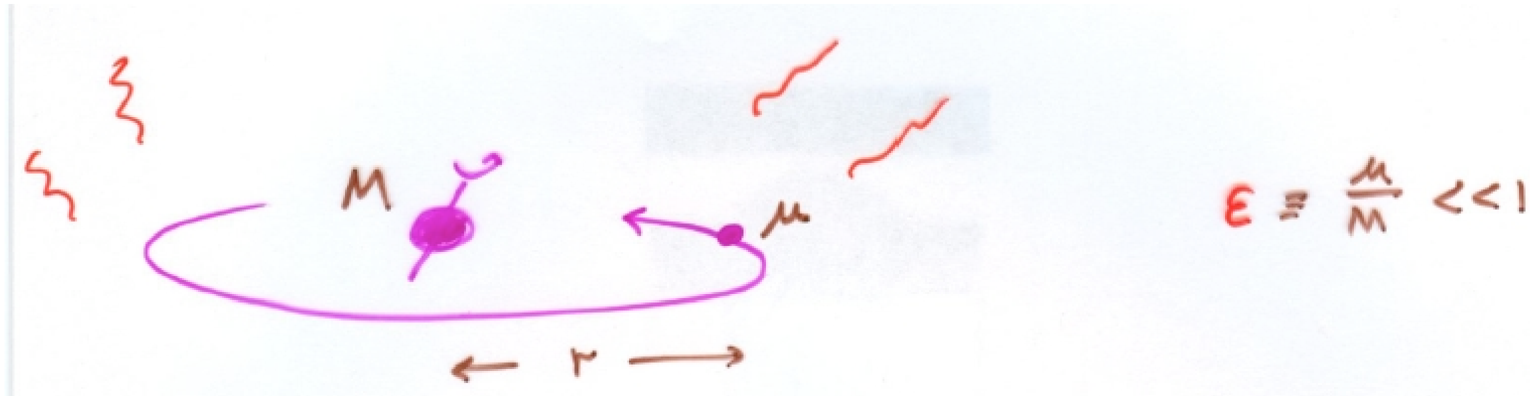
Extreme mass-ratio inspirals



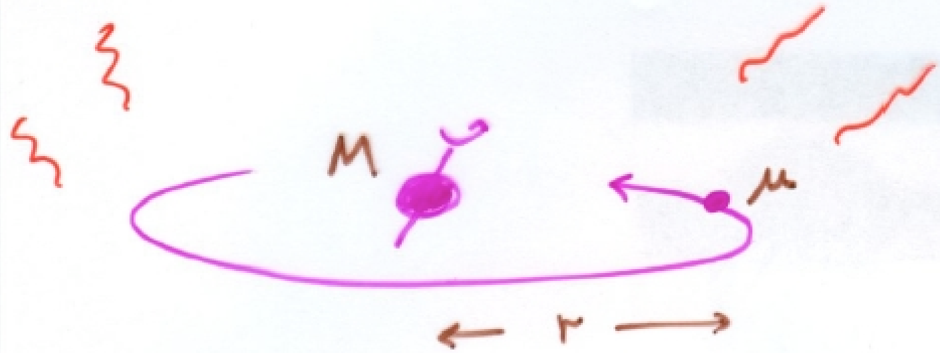
- Consider the one family of spacetimes generated by the circular inspiral of a small black hole into a larger black hole

$$g_{\alpha\beta} = g_{\alpha\beta}(x^\gamma, \varepsilon)$$

- As $\varepsilon \to 0$, what is the mathematical nature of the dependence on ε ?

- Two aspects: source or orbital motion (Tanja's talk), wave generation (this talk)

Timescales in the Problem



$$\varepsilon \equiv \frac{\mu}{M} << 1$$

$$\boxed{\tau_{orb} \sim M}\,, \quad E_{orb} \sim \mu\,, \quad \delta g_{ab} \sim \frac{\mu}{M}\,, \quad \delta \dot{g}_{ab} \sim \frac{\delta g_{ab}}{\tau_{orb}} \sim \frac{\mu}{M^2}$$

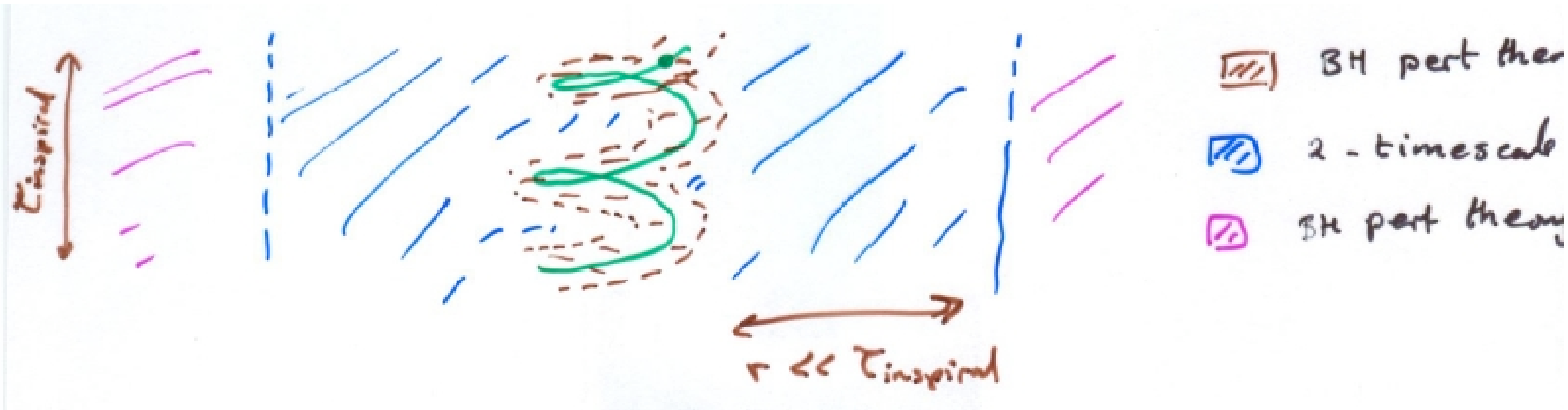
$$\dot{E} \sim M^2 (\delta \dot{g}_{ab})^2 \sim \frac{\mu^2}{M^2}\,, \quad \tau_{inspiral} \sim \frac{E}{\dot{E}} \sim \frac{M^2}{\mu} \sim \boxed{\frac{M}{\varepsilon} \sim \tau_{inspiral}}$$

$$N_{cycles} \sim \frac{\tau_{inspiral}}{\tau_{orb}} \sim \frac{1}{\varepsilon}$$

Orbital phase: $\phi(t) \sim \phi_0 + \omega_0 t + \dot{\omega}_0 t^2 \sim \phi_0 + \frac{t}{\tau_{orb}} + \underbrace{\frac{t^2}{\tau_{orb}\,\tau_{inspiral}}}_{\delta\phi}$

$\delta\phi \sim 1$ at $t \sim \boxed{\tau_{dephase} \sim \frac{M}{\sqrt{\varepsilon}}}$

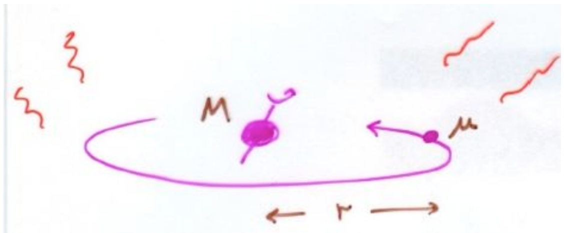
More precise version of question



- Different results for behavior of $g_{\alpha\beta}(\varepsilon)$ at distances from small black hole of order $\sim \varepsilon$, ~ 1, $\sim 1/\varepsilon$. Global consistent solution obtained by matching.

- Here focus on blue domain

Catalog of approximation schemes

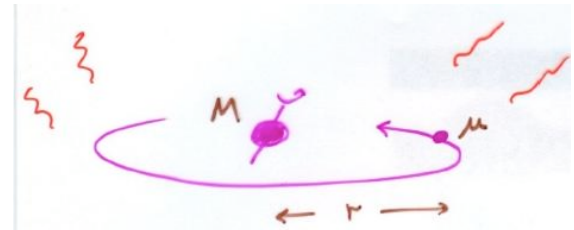


Minkowski → Linearized Perturbations → Second Order Perturbations

Kerr → Linearized Perturbations → Second Order Perturbations

$$
\begin{aligned}
M/R &\ll 1, \\
M/R &\ll R^2/T^2
\end{aligned}
$$

Catalog of approximation schemes



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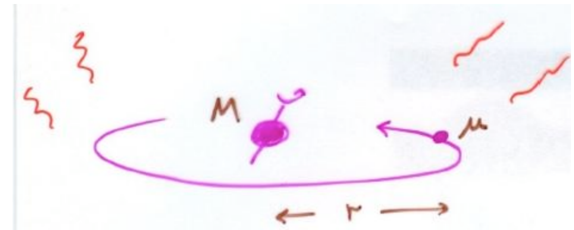

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Kerr → Linearized Perturbations → Second Order Perturbations

Kerr → Adiabatic → Post-1 Adiabatic

$$M/R \ll 1, \quad M/R \ll R^2/T^2$$

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Problems with conventional pert. scheme

1. To linear order, conservation of stress energy forces the particle to move on a geodesic
2. Waveforms computed from an inspiralling motion using linearized theory expected to be gauge dependent
3. Going to second order does not help: it breaks down after a dephasing time $\sim M/\sqrt{\varepsilon}$

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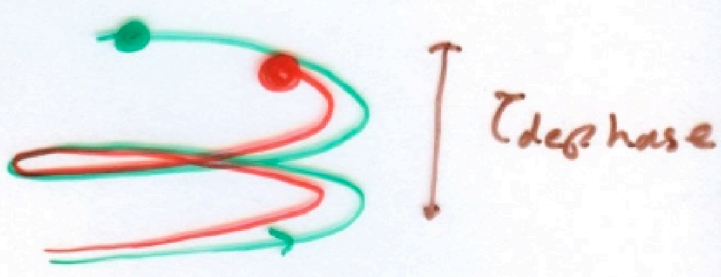
$$g_{ab} = g^{Kerr}_{ab} + \varepsilon h^{(1)}_{ab} + \varepsilon^2 h^{(2)}_{ab}$$

After dephasing time,

$$\varepsilon h^{(1)}_{ab} \sim \varepsilon^2 h^{(2)}_{ab}$$



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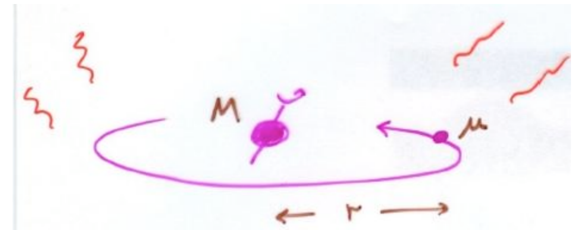

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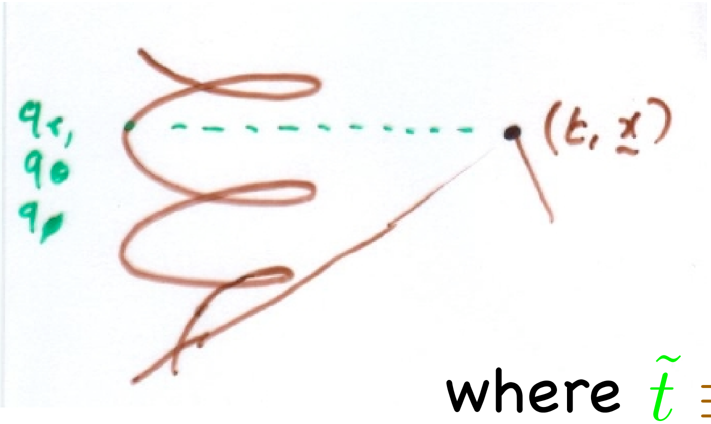
Metric ansatz in two timescale method

- Ansatz: $$g_{\alpha\beta}(t,r,\theta,\phi;\varepsilon) = g^{(0)}_{\alpha\beta}(r,\theta,\phi) + \varepsilon g^{(1)}_{\alpha\beta}(q_r,q_\theta,q_\phi,\tilde{t},r,\theta,\phi) + \varepsilon^2 g^{(2)}_{\alpha\beta}(q_r,q_\theta,q_\phi,\tilde{t},r,\theta,\phi) + \ldots$$



where $\tilde{t} \equiv \varepsilon t$ and periodicity in q's assumed

- Here the angle variables q_r, q_θ, q_ϕ are obtained by solving for the orbital motion, and are of the form

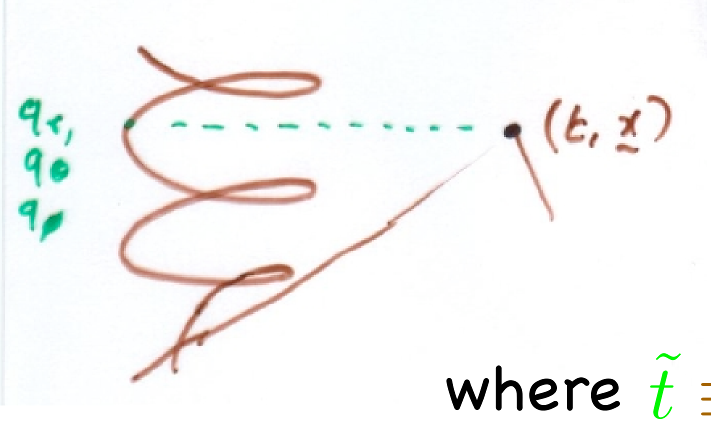
$$q_i(t,\varepsilon) = \frac{1}{\varepsilon} f_i^{(0)}(\varepsilon t) + f_i^{(1)}(\varepsilon t) + \ldots$$

- Self-consistency verified by substitution into Einsteins eqns.

Leading (adiabatic) order

- Ansatz:

$$
\begin{aligned}
g_{\alpha\beta}(t,r,\theta,\phi;\varepsilon) &= g^{(0)}_{\alpha\beta}(r,\theta,\phi) \\
&+\varepsilon g^{(1)}_{\alpha\beta}(q_r,q_\theta,q_\phi,\tilde{t},r,\theta,\phi) \\
&+\varepsilon^2 g^{(2)}_{\alpha\beta}(q_r,q_\theta,q_\phi,\tilde{t},r,\theta,\phi) \\
&+\dots
\end{aligned}
$$



where $\tilde{t} \equiv \varepsilon t$ and periodicity in q's assumed

- Obtain "linearized Einstein equation" $G^{(1,0)}_{\alpha\beta}[g^{(1)}_{\gamma\delta}] = 0$ as a PDE on a 6D manifold with coordinates $q_r, q_\theta, q_\phi, r, \theta, \phi$

- Solution is

$$
\begin{aligned}
g^{(1)}_{\alpha\beta} &= \frac{\partial g^{(0)}_{\alpha\beta}}{\partial M}\delta M(\tilde{t}) + \frac{\partial g^{(0)}_{\alpha\beta}}{\partial a}\delta a(\tilde{t}) + \dots \\
&+F_{\alpha\beta}[q_r,q_\theta,q_\phi,r,\theta,\phi,E(\tilde{t}),L_z(\tilde{t}),K(\tilde{t})]
\end{aligned}
$$

where the function $F_{\alpha\beta}$ is the same as in standard pert theory with geodesic orbits

Conclusions

- The two-timescale method gives a self-consistent framework for computing extreme mass ratio inspirals that resolves the difficulties with standard perturbation theory