Strongly Hyperbolic Extensions of the ADM Hamiltonian

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General relativity is derived from a variational principle:

$$S = \int d^4x \sqrt{-g}\mathcal{R}$$

and has a Hamiltonian formulation:

$$H = \int d^3x \left\{ \alpha\mathcal{H} + \beta^a \mathcal{M}_a \right\}$$

The action and Hamiltonian

- shape the way we think about the theory
- lead to physical/mathematical insights
- serve as foundations for analytical and computational techniques.

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Goal: Extend the ADM action/Hamiltonian in a way that is suitable for numerical calculations.

Key requirements for numerical relativity:

- Gauge conditions are best determined by evolution equations for the lapse and shift.
- Evolution equations must be (at least) strongly hyperbolic.

(ADM with prescribed lapse and shift is weakly hyperbolic.)

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Goal: Extend the ADM action/Hamiltonian in a way that

- Lagrange's/Hamilton's equations include evolution equations for the lapse and shift.
- Lagrange's/Hamilton's equations are (at least) strongly hyperbolic

How To

ADM action:

$$S = \int dt \int d^3x \left\{ P^{ab}\dot{g}_{ab} - \alpha\mathcal{H} - \beta^a\mathcal{M}_a \right\}$$

Define momenta conjugate to lapse and shift:

$$\begin{aligned} \pi &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = 0 \\ \rho_a &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\beta}^a} = 0 \end{aligned}$$

This leads to constraints $\pi = 0$ and $\rho_a = 0$.

Introduce undetermined multipliers Λ and Ω^a. The action becomes:

$$S = \int dt \int d^3x \left\{ P^{ab}\dot{g}_{ab} + \pi\dot{\alpha} + \rho_a\dot{\beta}^a - \alpha\mathcal{H} - \beta^a\mathcal{M}_a - \Lambda\pi - \Omega^a\rho_a \right\}$$

Key observation: The variational principle is unchanged if we allow Λ and Ω^a to depend on the canonical variables:

$$\begin{aligned} \Lambda &\rightarrow \Lambda + \hat{\Lambda} \\ \Omega^a &\rightarrow \Omega^a + \hat{\Omega}^a \end{aligned}$$

$\hat{\Lambda}$ and $\hat{\Omega}^a$ are linear functions of p, $\partial_a q$, and 1 with coefficients that depend on the q's.

The extended theory is described by the action:

$$\begin{aligned} S &= \int dt \int d^3x \Big\{ P^{ab}\dot{g}_{ab} + \pi\dot{\alpha} + \rho_a\dot{\beta}^a - \alpha\mathcal{H} - \beta^a\mathcal{M}_a \\ &\qquad -(\Lambda + \hat{\Lambda})\pi - (\Omega^a + \hat{\Omega}^a)\rho_a \Big\} \end{aligned}$$

or equivalently by the Hamiltonian

$$H = \int d^3x \Big\{ \alpha\mathcal{H} + \beta^a\mathcal{M}_a + (\Lambda + \hat{\Lambda})\pi + (\Omega^a + \hat{\Omega}^a)\rho_a \Big\}$$

with first class constraints:

$$\begin{aligned} \pi &= 0 \\ \rho_a &= 0 \\ \mathcal{H} &= 0 \\ \mathcal{M}_a &= 0 \end{aligned}$$

Evolution equations of motion:

$$
\begin{aligned}
\dot{g}_{ab} &= (\text{usual ADM}) + \frac{\partial\hat{\Lambda}}{\partial P^{ab}}\pi + \frac{\partial\hat{\Omega}^c}{\partial P^{ab}}\rho_c \\
\dot{P}^{ab} &= (\text{usual ADM}) - \frac{\partial\hat{\Lambda}}{\partial g_{ab}}\pi - \frac{\partial\hat{\Omega}^c}{\partial g_{ab}}\rho_c \\
&\qquad + \partial_d\left(\frac{\partial\hat{\Lambda}}{\partial(\partial_d g_{ab})}\pi\right) + \partial_d\left(\frac{\partial\hat{\Omega}^c}{\partial(\partial_d g_{ab})}\rho_c\right) \\
\dot{\alpha} &= \Lambda + \hat{\Lambda} + \frac{\partial\hat{\Lambda}}{\partial\pi}\pi + \frac{\partial\hat{\Omega}^c}{\partial\pi}\rho_c \\
\dot{\beta}^a &= \Omega^a + \hat{\Omega}^a + \frac{\partial\hat{\Lambda}}{\partial\rho_a}\pi + \frac{\partial\hat{\Omega}^c}{\partial\rho_a}\rho_c \\
\dot{\pi} &= \ldots \\
\dot{\rho}_a &= \ldots
\end{aligned}
$$

Example:

$$\begin{aligned}
\hat{\Lambda} &= (\dot{\tilde{\alpha}}/\tilde{\alpha})\alpha + \beta^a D_a \alpha - \frac{\alpha^2}{2\sqrt{g}} P + \frac{\alpha^3}{8\sqrt{g}} \pi \\
\hat{\Omega}^a &= (\dot{\tilde{\alpha}}/\tilde{\alpha})\beta^a + \beta^b \tilde{D}_b \beta^a + \alpha^2 (\Gamma^a_{bc} - \tilde{\Gamma}^a_{bc}) g^{bc} \\
&\quad -\alpha D^a \alpha - \frac{\alpha^3}{2\sqrt{g}} \rho^a
\end{aligned}$$

where $\tilde{\alpha}$ and $\tilde{\Gamma}^a_{bc}$ are functions of the spacetime coordinates (their transformation properties insure spatial covariance and time reparametrization invariance)

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The equations of motion for g_{ab}, α, β^a, *etc* are:

- strongly hyperbolic with physical characteristics
- equivalent in their principal parts to a 3+1 splitting of the generalized harmonic equations

Conclusion:

The procedure outlined here allows one to

- introduce dynamical gauge conditions for the lapse and shift
- change the level of hyperbolicity of the evolution system

while maintaining the variational and Hamiltonian structures of the theory.

Details: arXiv:0803.0334 [gr-qc]