Post-Newtonian Theory and Gravitational Wave Physics

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Dynamical Horizons, Binary Coalescences, Simulations and Waveforms

Penn State

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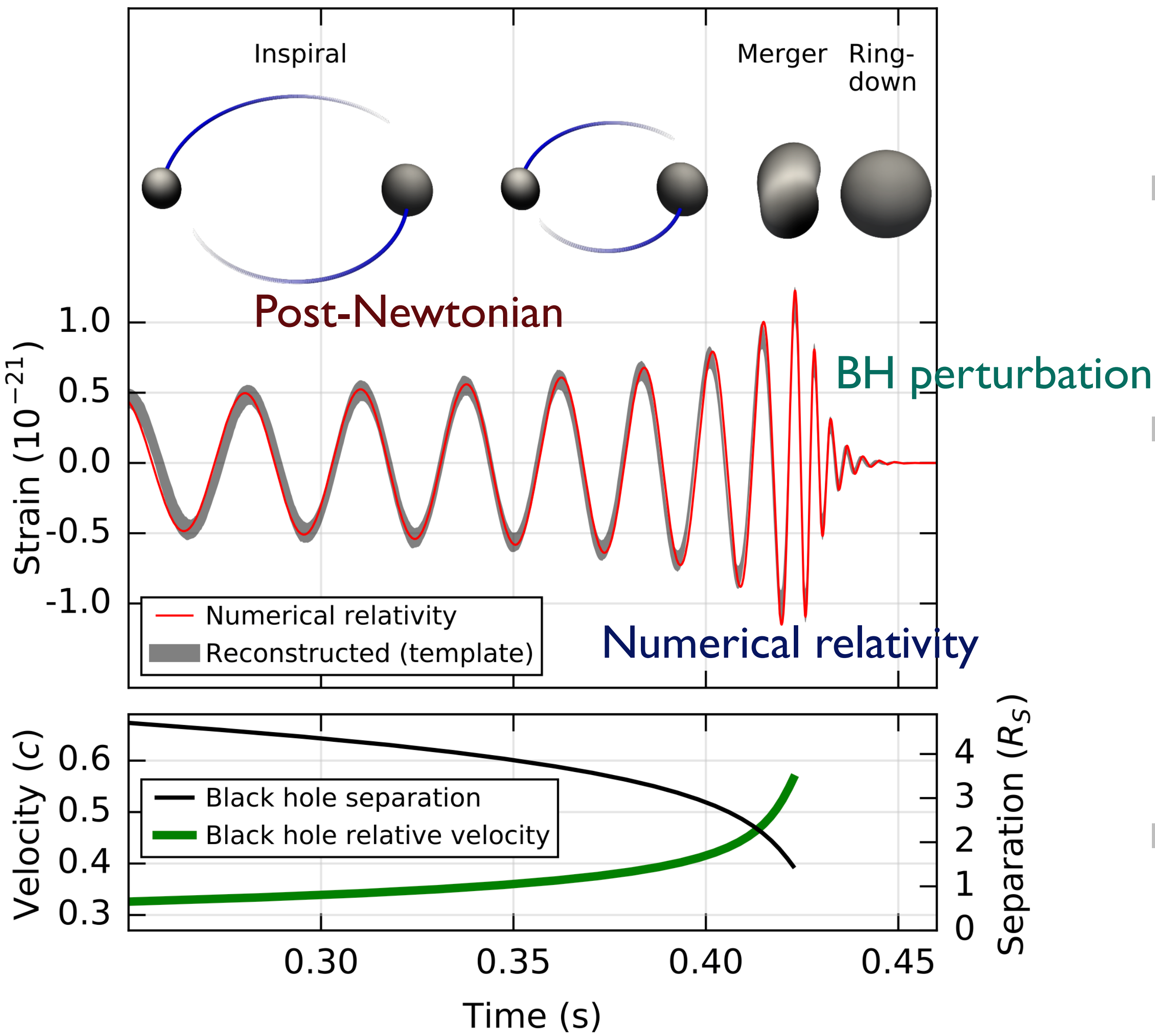
Post-Newtonian model of compact binary inspiral



LSC + Virgo, Phys. Rev. Lett. 116, 061102 (2016)

Post-Newtonian Theory

- Solving the two body problem in GR using perturbation theory.
- Uses various approximations to model the dynamics and deduce the gravitational waveforms from the compact binaries.
- Goal: Construct very accurate representation of the phase and amplitude of the gravitational wave signal.

Post-Newtonian theory

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

$$\Box h_{\mu\nu} = 16\,\pi\,T_{\mu\nu} + \mathcal{F}(h,h)$$

$$h_{\alpha\beta} = \epsilon\,h^{(1)}_{\alpha\beta} + \epsilon^2\,h^{(2)}_{\alpha\beta} + \cdots$$

$$T_{\alpha\beta} = \epsilon\,T^{(1)}_{\alpha\beta} + \epsilon^2\,T^{(2)}_{\alpha\beta} + \cdots$$

Applicable to
slowly moving, weakly stressed,
weakly self-gravitating systems

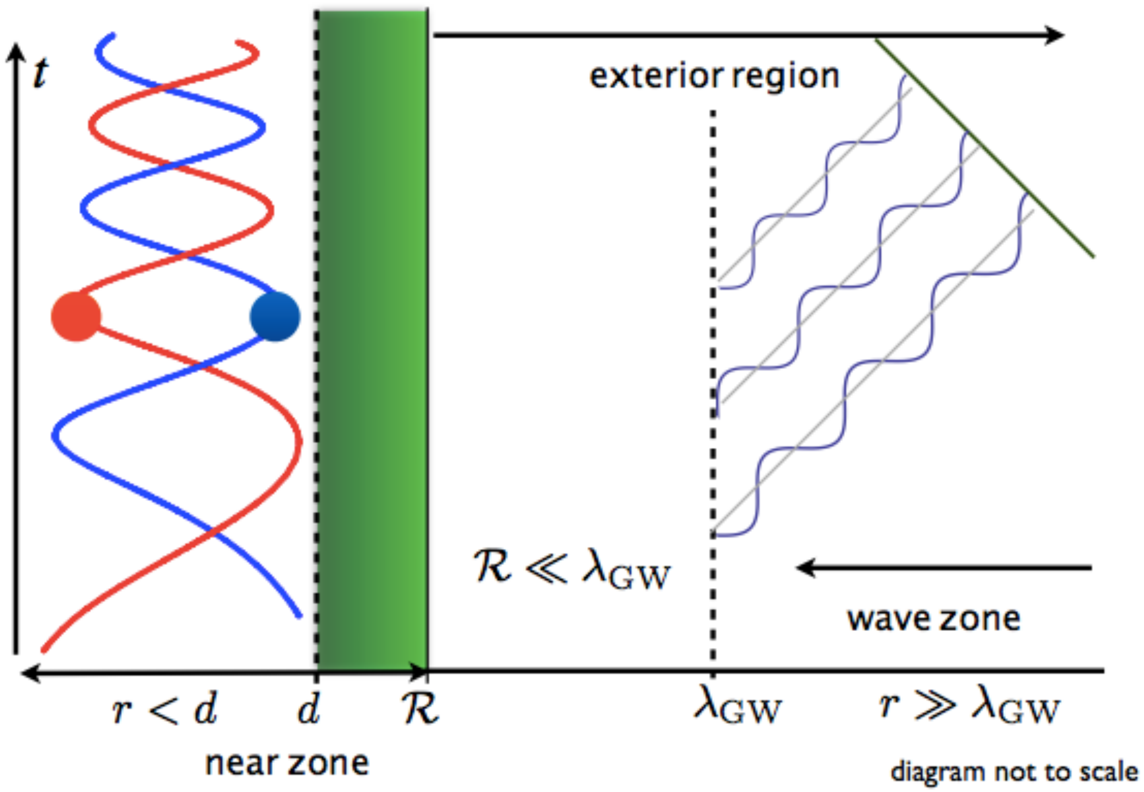
$$|T^{0i}/T^{00}| \sim \sqrt{|T^{ij}/T^{00}|} \sim \sqrt{|U/c^2|} \ll 1$$

Solving order by order

$$\Box h^{(1)}_{\mu\nu} = 16\,\pi\,T^{(1)}_{\mu\nu}$$

$$\Box h^{(2)}_{\mu\nu} = 16\,\pi\,T^{(2)}_{\mu\nu} + \mathcal{F}(h^{(1)}, h^{(1)})$$

Application to compact binaries



Buonanno, Sathyaprakash 2014

Multipole Expansion in the exterior region

Post-Newtonian expansion in the near zone

Matching of these two expansions in the exterior part of the near zone.

Gravitational Waveform

$$h_{km}^{TT} = \frac{2G}{c^4 R}\mathcal{P}_{ijkm}\Big\{U_{ij} + \frac{1}{c}\left[\frac{1}{3}N_a U_{ija} + \frac{4}{3}\varepsilon_{ab(i}V_{j)a}N_b\right]$$
$$+ \frac{1}{c^2}\left[\frac{1}{12}N_{ab}U_{ijab} + \frac{1}{2}\varepsilon_{ab(i}V_{j)ac}N_{bc}\right]$$
$$+ \frac{1}{c^3}\left[\frac{1}{60}N_{abc}U_{ijabc} + \frac{2}{15}\varepsilon_{ab(i}V_{j)acd}N_{bcd}\right]$$
$$+ \frac{1}{c^4}\left[\frac{1}{360}N_{abcd}U_{ijabcd} + \frac{1}{36}\varepsilon_{ab(i}V_{j)acde}N_{bcde}\right] + O(\varepsilon^5)\Big\}\ .$$

$$\mathcal{P}_{ijkm}(\mathbf{N}) = (\delta_{ik} - N_i N_k)(\delta_{jm} - N_j N_m) - \frac{1}{2}(\delta_{ij} - N_i N_j)(\delta_{km} - N_k N_m)\ .$$

Thorne, 1986

$U_L \longrightarrow$ Mass-type radiative multipole

$V_L \longrightarrow$ Current type radiative multipole

Energy flux

$$\frac{dE_B}{dT_R} = -\frac{c^3}{32\pi G}\int\left(\frac{\partial h_{ij}^{TT}}{\partial T_R}\right)^2 R^2 d\Omega(\mathbf{N}) \ .$$

At the 2PN approximation this yields (with $U^{(n)} \equiv d^n U/dT_R^n$)

$$\frac{dE_B}{dT_R} = -\frac{G}{c^5}\left\{\frac{1}{5}U_{ij}^{(1)}U_{ij}^{(1)} + \frac{1}{c^2}\left[\frac{1}{189}U_{ijk}^{(1)}U_{ijk}^{(1)} + \frac{16}{45}V_{ij}^{(1)}V_{ij}^{(1)}\right]\right.$$
$$\left. + \frac{1}{c^4}\left[\frac{1}{9072}U_{ijkm}^{(1)}U_{ijkm}^{(1)} + \frac{1}{84}V_{ijk}^{(1)}V_{ijk}^{(1)}\right] + O(\varepsilon^6)\right\} \ .$$

Relating radiative multipoles to source

Instantaneous Hereditary Source multipole (Mass type)

$$U_{ij}(T_R) = I^{(2)}_{ij}(T_R) + \frac{2Gm}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b}\right) + \frac{11}{12}\right] I^{(4)}_{ij}(T_R - \tau) + O(\varepsilon^5)\ , \qquad (2.6a)$$

$$U_{ijk}(T_R) = I^{(3)}_{ijk}(T_R) + \frac{2Gm}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b}\right) + \frac{97}{60}\right] I^{(5)}_{ijk}(T_R - \tau) + O(\varepsilon^5)\ , \qquad (2.6b)$$

$$V_{ij}(T_R) = J^{(2)}_{ij}(T_R) + \frac{2Gm}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{\tau}{2b}\right) + \frac{7}{6}\right] J^{(4)}_{ij}(T_R - \tau) + O(\varepsilon^4)\ , \qquad (2.6c)$$

Source multipole (Current type)

Various nonlinear effects caused by multipole interactions appear at different post-Newtonian orders.

The matching procedure is central
to expressing the radiative multipoles
in terms of the source multipoles

Various Nonlinear effects

$$
\begin{aligned}
U_{ij}(U) =& I^{(2)}_{ij}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{2}\right] I^{(4)}_{ij}(U-\tau) \\
&+ \frac{G}{c^5}\left\{ -\frac{2}{7}\int_0^{+\infty} d\tau I^{(3)}_{a\langle i}(U-\tau) I^{(3)}_{j\rangle a}(U-\tau) \right. \\
&+ \frac{1}{7} I^{(5)}_{a\langle i} I_{j\rangle a} - \frac{5}{7} I^{(4)}_{a\langle i} I^{(1)}_{j\rangle a} - \frac{2}{7} I^{(3)}_{a\langle i} I^{(2)}_{j\rangle a} + \frac{1}{3}\varepsilon_{ab\langle i} I^{(4)}_{j\rangle a} J_b \\
&\left. + 4\left[W^{(2)} I_{ij} - W^{(1)} I^{(1)}_{ij}\right]^{(2)} \right\} + 2\left(\frac{GM}{c^3}\right)^2 \\
&\times \int_0^{+\infty} d\tau I^{(5)}_{ij}(U-\tau) \left[\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right) \right. \\
&\left. + \frac{124\,627}{44\,100}\right] + \mathcal{O}(7),
\end{aligned}
\tag{2.2}
$$

Tails

Memory

Tails of Tails

Blanchet, '98.

PN equations of motion

$$\mathbf{v} \equiv \frac{d\mathbf{x}}{dt} \, ,$$

$$\mathbf{a} \equiv \frac{d\mathbf{v}}{dt} \equiv \frac{d^2\mathbf{x}}{dt^2} = -\omega_{\rm 2PN}^2\,\mathbf{x} + O(\varepsilon^5) \, .$$

$$\omega_{\rm 2PN}^2 \equiv \frac{Gm}{r^3}\left[1 - (3-\nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2\right)\gamma^2\right]$$

$$\gamma = \frac{Gm}{rc^2}$$

Energy balance and GW phase

$$\boxed{\frac{d}{dt}E_{\mathrm{orb}} = \mathcal{F}}$$

- Orbital Energy:

0PN 1PN 2PN

$$\mathcal{E} = -\frac{c^2}{2}\nu m x\left\{1 - \frac{1}{12}(9+\nu)x - \frac{1}{8}\left(27 - 19\nu + \frac{\nu^2}{3}\right)x^2\right\}$$

- Energy Flux:

$$\frac{dE_B}{dT_R} = -\frac{32c^5}{5G}\nu^2 x^5\left\{1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi\, x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2\right\}$$

$$x = \left(\frac{Gm\,\omega}{c^3}\right)^{2/3}$$

Blanchet, Damour, Iyer, 1995

Modelling the GW phasing

Given PN expressions forGW flux and Orbital energy,
how do we construct the phasing?

Differential Form

$$\frac{d\phi}{dt} - \frac{v^3}{M} = 0,$$
$$\frac{dv}{dt} + \frac{\mathcal{F}(v)}{ME'(v)} = 0,$$

Integral Form

$$t(v) = t_{\rm ref} + M \int_v^{v_{\rm ref}} dv \frac{E'(v)}{\mathcal{F}(v)},$$
$$\phi(v) = \phi_{\rm ref} + \int_v^{v_{\rm ref}} dv v^3 \frac{E'(v)}{\mathcal{F}(v)},$$

Different approximants arise depending
on whether integral or differential form is used
and how the fraction $\frac{E'(v)}{\mathcal{F}(v)}$ is treated.

TaylorT1 Approximant

Use the PN expressions for
energy and flux and solve the differential equations numerically
using appropriate initial conditions.

$$\frac{d\phi^{(\mathrm{T1})}}{dt} - \frac{v^3}{M} = 0,$$

$$\frac{dv}{dt} + \frac{\mathcal{F}(v)}{ME'(v)} = 0.$$

TaylorT4 Approximant

Very similar to TaylorT1, but the
ratio of the polynomials to the consistent PN order
and perform the numerical integration.

Gives better agreement with
NR waveforms.

TaylorT2 Approximant

Uses integral form, truncates the ratio at the consistent PN order and then integrate the two expressions.

$$\phi_{n/2}^{(\mathrm{T2})}(v) = \phi_{\mathrm{ref}}^{(\mathrm{T2})} + \phi_N^v(v) \sum_{k=0}^{n} \hat{\phi}_k^v v^k,$$

$$t_{n/2}^{(\mathrm{T2})}(v) = t_{\mathrm{ref}}^{(\mathrm{T2})} + t_N^v(v) \sum_{k=0}^{n} \hat{t}_k^v v^k.$$

Phase evolution involves solving a pair of transcendental equations and is computationally expensive

TaylorF2 Approximant

Uses stationary phase approximation.

$$\tilde{h}^{\text{spa}}(f) = \frac{a(t_f)}{\sqrt{\dot{F}(t_f)}} e^{i[\psi_f(t_f) - \pi/4]},$$

$$\psi_f(t) \equiv 2\pi f t - 2\phi(t),$$

$$t_f = t_{\text{ref}} + M \int_{v_f}^{v_{\text{ref}}} \frac{E'(v)}{\mathcal{F}(v)} dv, \tag{3.15a}$$

$$\psi_f(t_f) = 2\pi f t_{\text{ref}} - \phi_{\text{ref}} + 2 \int_{v_f}^{v_{\text{ref}}} (v_f^3 - v^3) \frac{E'(v)}{\mathcal{F}(v)} dv. \tag{3.15b}$$

Very widely used as it is in frequency domain

Frequency Domain GW Phasing

Using Stationary Phase Approximation

$$\psi_f(t_f) = 2\pi f t_{\rm ref} - \phi_{\rm ref} + 2\int_{v_f}^{v_{\rm ref}} (v_f^3 - v^3)\frac{E'(v)}{\mathcal{F}(v)}dv.$$

$$\eta = \frac{m_1 m_2}{(m_1+m_2)^2}$$

Structure of the Phasing formula

$$v = (\pi M f)^{1/3}$$

Coalescence phase

$$\Psi(f) = 2\pi f\, t_c - \phi_c + \frac{3}{128\eta\, v^5}\left[\sum_{k=0}^{N} \Psi_k\, v^{k-5}\right]$$

Coalescence time

tidal deformability parameters

masses spins

$$\Psi(f) = \Psi(f; m_1, m_2, \chi_1, \chi_2, e, \lambda_1, \lambda_2, \cdots)$$

eccentricity

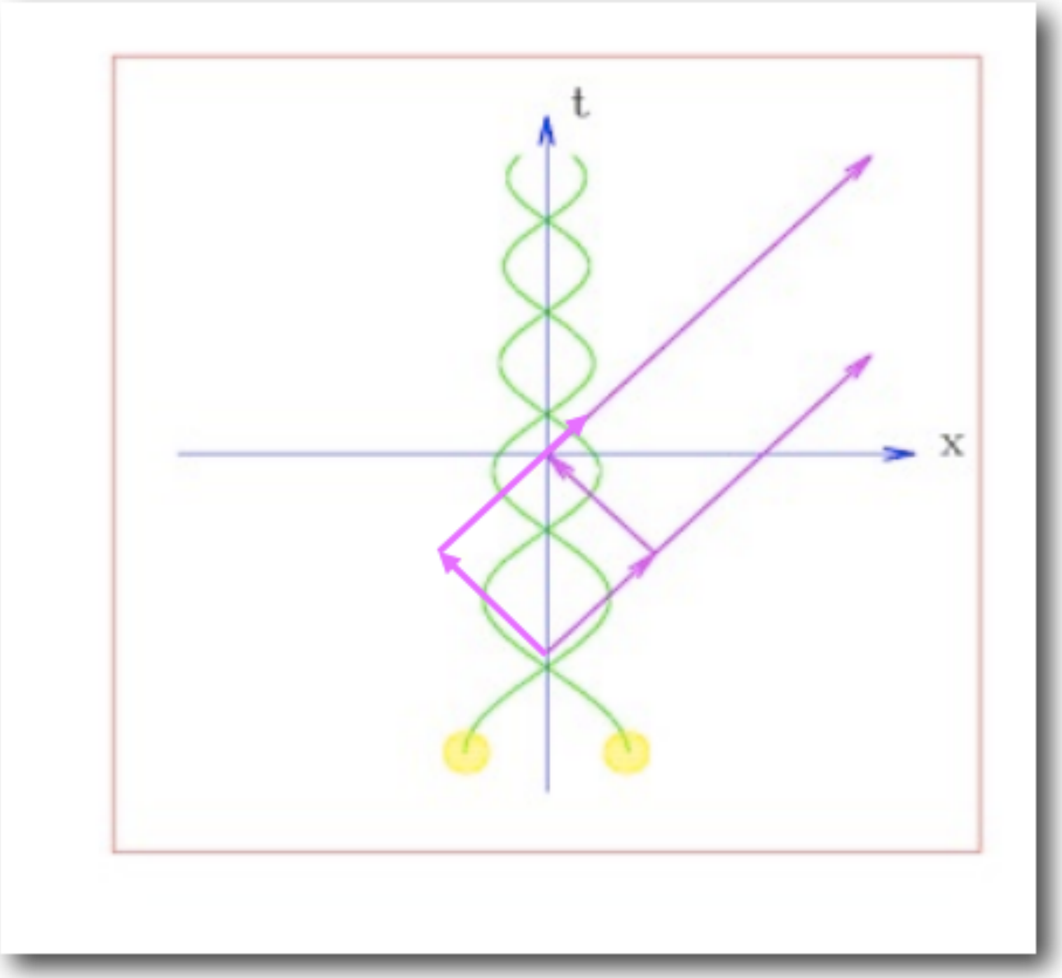
Table of Physical effects: Quick glance

PN Order	Effect
0PN	Chirp Mass
1PN	Possibility to measure component masses, Periastron Advance,
1.5PN	Tails of GWs, Spin-orbit interaction
2PN	Spin-spin interaction, spin-induced quadrupole
2.5PN	Black hole Horizon Flux (spinning)
3PN	Tails of Tails, Tail^2
3.5PN	Spin-induced octupole
4PN	Black hole Horizon Flux (nonspinning)
5PN	Tidal interactions

Hereditary effects

- Contributions which depend on the dynamics if the system in the past (Vs instantaneous which is function of the retarded time).
- Tails: Due to back-scattering of GWs by the background space time.
- Tails of Tails: Tails being scattered by background curvature.
- Memory: Re-radiation of stress-energy tensor of the linear GWs.

Gravitational wave tails



Blanchet and Schaefer (1994)

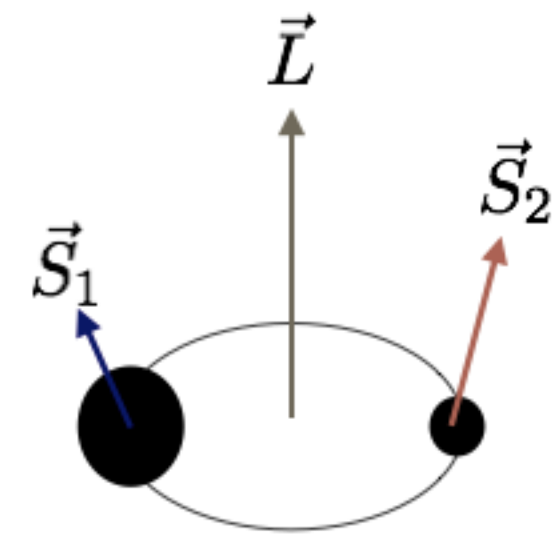
Computation of Hereditary terms:

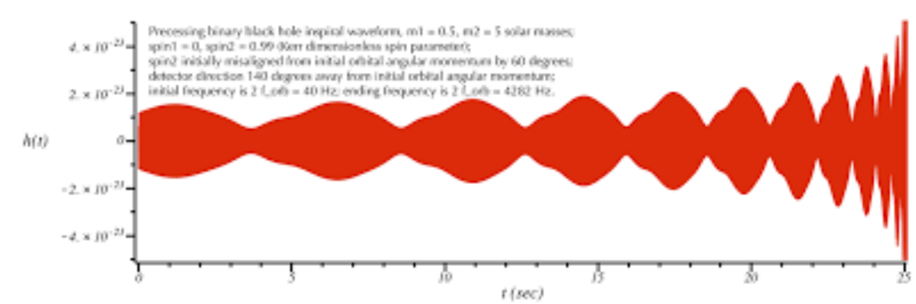
Blanchet, Damour, Iyer, Schafer, Will, Wiseman, KGA, Qusailah, Favata, Sinha, Mishra, …

Spin effects

Clues to formation channels of BBHs

- Spin-orbit interaction (there can be spin-orbit resonances).
- Spin-spin interactions (carry information about spin-induced multipoles of the compact binaries)
- When spins are arbitrarily aligned w.r.t orbital angular momentum, precession can lead to complicated modulations in the waveform.
- Many of these carry unique imprints about the formation scenario of the binary black holes.





soundsofspacetime.org

Spin-induced multipoles

Mass-type multipoles

Mass

$\chi = |\mathbf{S}|/M^2$

$$Q_l + iS_l = M^{l+1}(i\chi)^l$$ ← Dimensionless spin

Current type multipoles

BH mimickers

Mass quadrupole of a compact object

$$Q_2 = -\kappa\, M^3\chi^2, \text{with } \kappa_{\rm BH} \equiv 1$$

$$\kappa_{\rm NS} \simeq 2 - 14$$
$$\kappa_{\rm BS} \simeq 10 - 150$$
$$\kappa_{\rm GS} \simeq +/-$$

Spin octupole of a compact object

$$S_3 = -\lambda\, M^4\,\chi^3 \text{ with } \lambda_{\rm BH} \equiv 1$$

$$\lambda_{\rm NS} \sim 4 - 30$$
$$\lambda_{\rm BS} \sim 10 - 200$$

$$\Psi_{\rm GW}(\kappa, \lambda)$$

Tidal interaction

Inferring Equation of State of compact objects

- Tidal interaction is a very unique aspect of the late time dynamics of compact binaries and carry signatures of the Equation of State of the compact object.
- Used to obtain the EoS constraints from GW170817.
- They start appearing at 5PN in phasing.
- Can be handy to test whether the observed system is a binary Black hole or not.

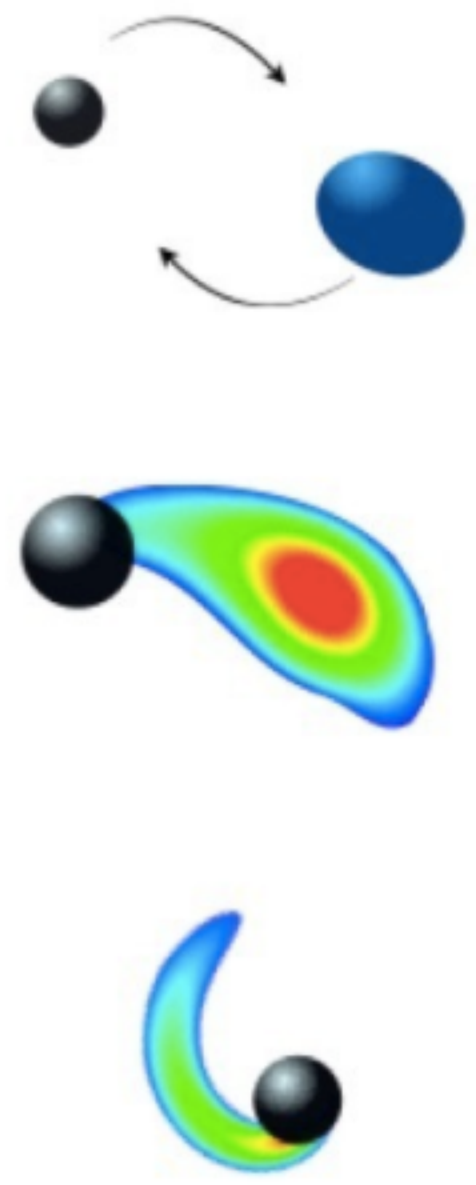


Fig: AEI, Potsdam

Orbital Eccentricity

Another way to track the binary formation

- GW emission is expected to circularise the binary.
- Residual eccentricity may carry important information about binary formation channel.

- GW Flux and polarisation are computed up to 3PN. [KGA, Blanchet, Damour, Gopakumar, Iyer, Mishra, Quasailah, Sinha…]
- Frequency domain representations are also available [KGA, Yunes, Berti, Will, Gopakumar, Haney, Kapadia, Huerta, Favata, Moore,…]

State of art

	No Spin	Spin-Linear	Spin-Squared	Tidal
Conservative Dynamics	4PN[a] [121, 122, 133] [126, 158–164]	3.5PN [52, 54, 141] [140, 165–169]	3PN [52, 54, 138] [137, 170–172]	7PN[b] [155–157]
Energy Flux at Infinity	3.5PN [95, 173, 174]	4PN [175–178]	3PN [53, 54, 179–181]	6PN [182]
Waveform Phase[c]	3.5PN [190]	4PN [175, 177, 178]	3PN [54, 179–181, 191]	6PN [182, 192]
Waveform Amplitude[e]	3PN[d] [194–197]	2PN [191, 198]	2PN [53, 54, 191, 198]	6PN [156, 182]
BH Horizon Energy Flux[g]	5PN [199]	3.5PN [200, 201]	4PN[f] [200, 201]	– –

Buonanno, Sathyaprakash 2014

Testing PN theory

Testing Tail effects

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Detecting a Tail Effect in Gravitational-Wave Experiments

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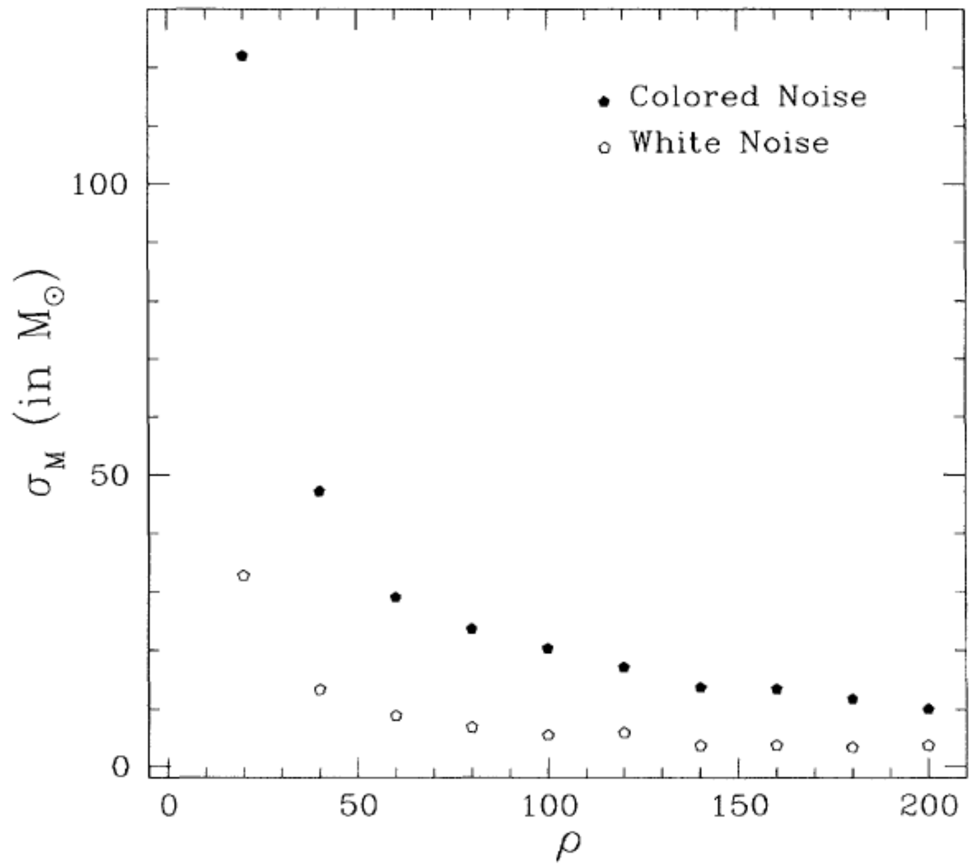
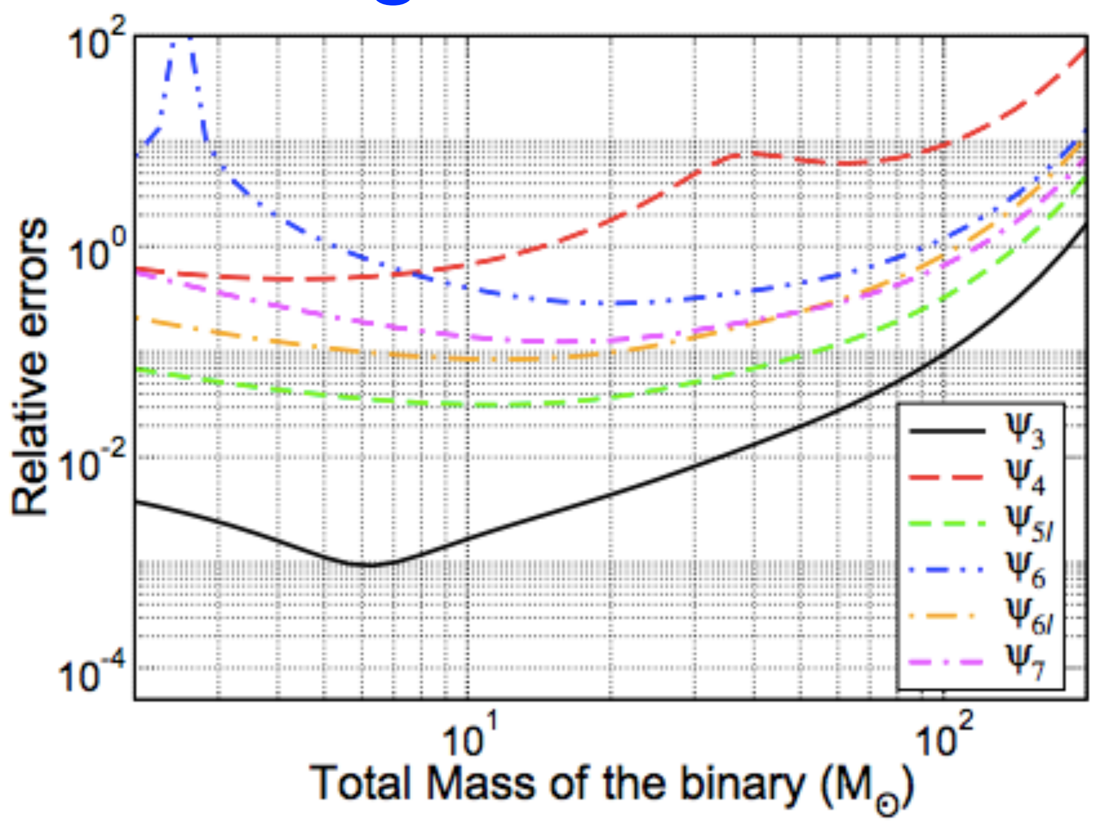
(Received 23 December 1993; revised manuscript received 7 October 1994)



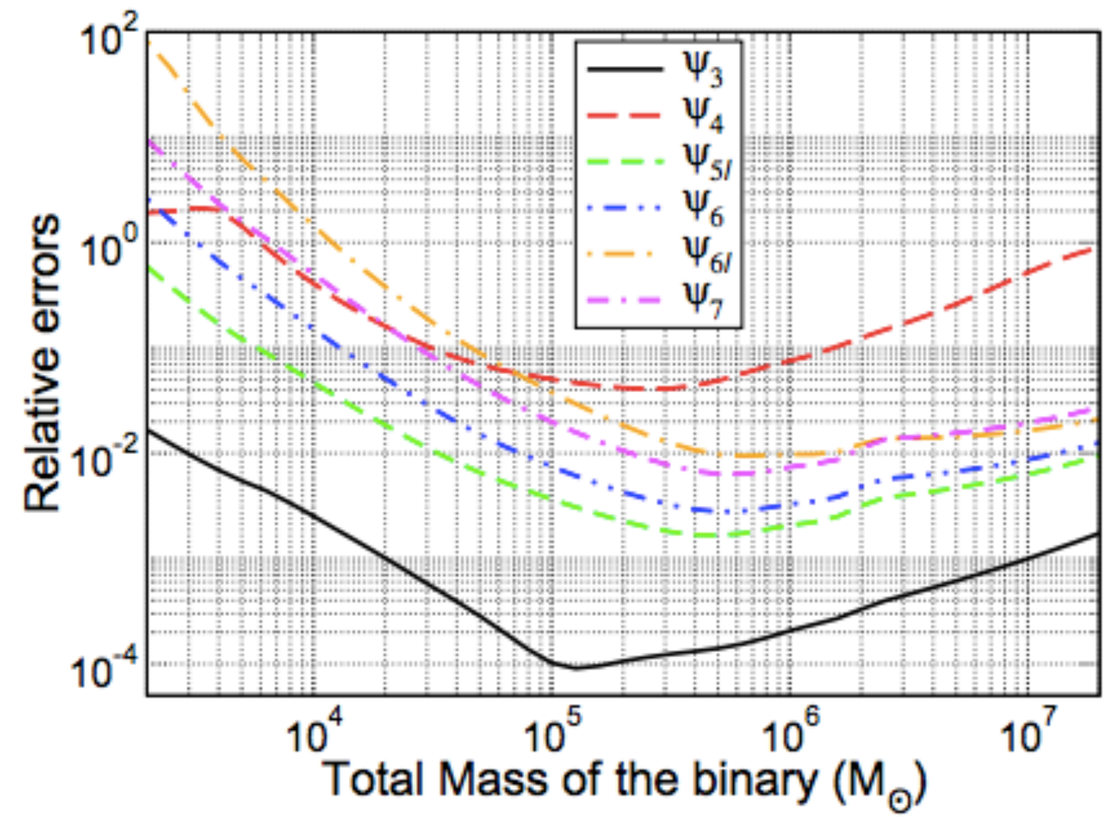
Parametrised Tests of PN theory

$$\Psi(f) = 2\pi f t_c - \Phi_c + \sum_{k=0}^{7} \left[\psi_k + \psi_{kl} \ln f\right] f^{(k-5)/3}.$$

3G ground-based

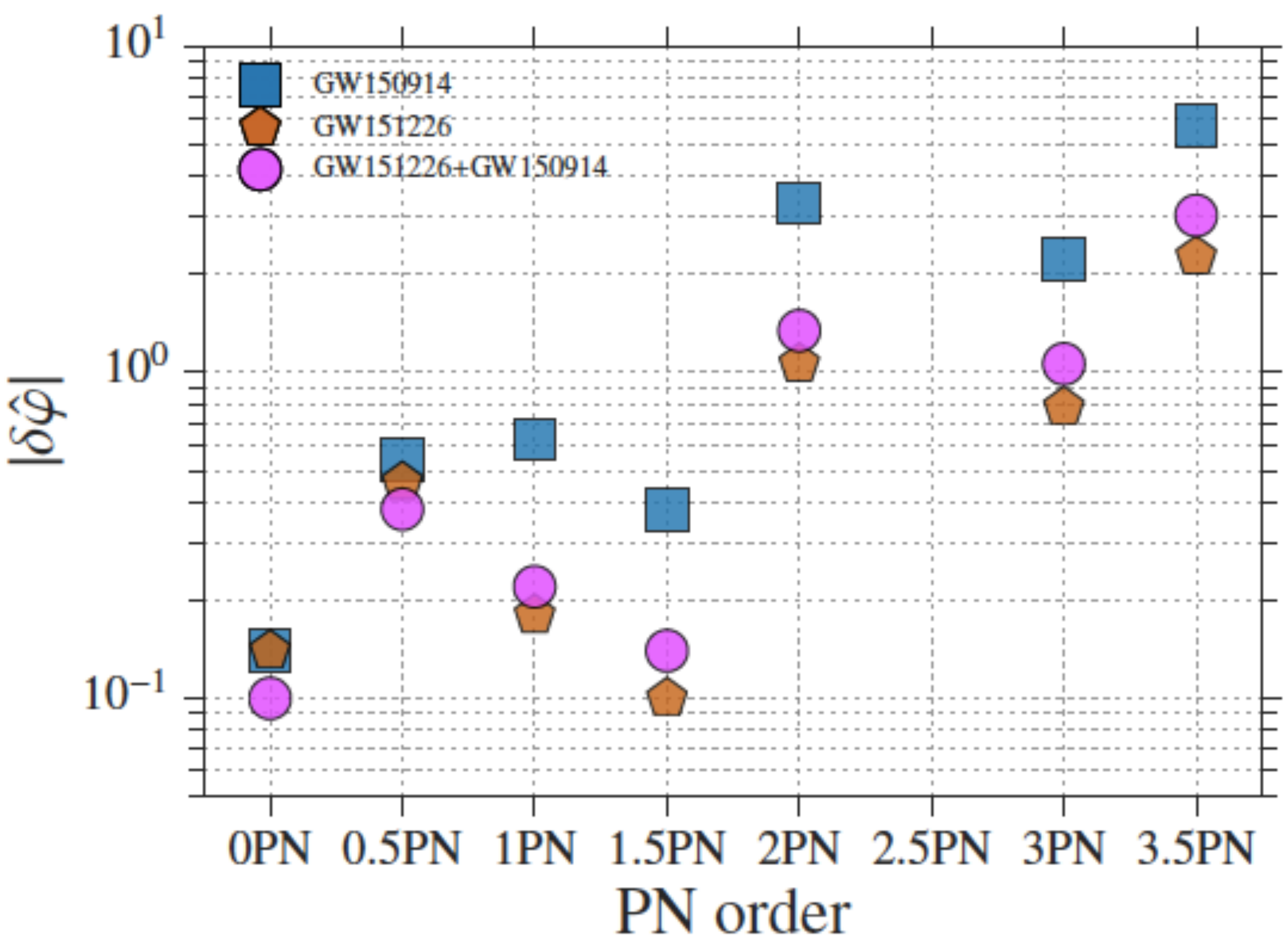


LISA



KGA, Iyer, Qusailah, Sathyaprakash, 2006

Bounds from the two GW events



LSC+Virgo, 2016, Phys. Rev. X 6, 041015 (2016)

$$\Psi_k \to \Psi_k^{\mathrm{GR}}(1+\delta\chi_k)$$

This is the current Observational Limit on the deviations allowed on the PN coefficients.

- Combines the two events during First observation run.

Open problems

- Higher order modelling of compact binaries.
- Modelling finite size effects to higher orders.
- Modelling systems with eccentricity and spins.

Summary

- Post-Newtonian theory has been very successful in modelling the compact binary dynamics and elegantly captures the highly nonlinear evolution of the binary.
- It has also paved the way for developing the waveform families such as Effective One Body and Phenomenological waveforms.
- These results have even widely used in inferring astrophysics and carrying out strong-field tests of gravity.