

## The Newtonian Limit

**Motivation:** linearization, a clear concept. Newtonian and post-Newtonian expansions are more of an art.

The relation between Newton's and Einstein's theories of gravity was clarified by Jürgen Ehlers, however, only recently mathematical results on the Newtonian limit and p-N-expansions have been obtained by Todd Oliynyk.

The purpose of this talk is to explain these results.

The plan of the talk:

- Ehlers frame theory ( Ehlers, 1981)

J. Ehlers, “On limit relations between, and approximative explanations of, physical theories”. *In: R. Barcan Marcus, G.J.W. Dorn, and P. Weingartner, Logic, methodology and philosophy of science VII, North Holland, Amsterdam, 1986, 387-403.*

- early results (Lottermoser,1990, Heilig, 1993, Rendall, 1994)
- new results ( Oliynyk, 2006)

## Frame Theory

The mathematical structures of Newton's and Einstein's theories are rather different:

**Newton:**

flat, 3-dim Euklidian space, absolute time,  
gravitational potential

**Einstein:**

4-dim spacetime, field equations from  
geometrical objects

Building on earlier work by Cartan, Friedrichs, Ehlers found a formulation which combines both theories!

Consider the following collection of objects:

$$(M^4, \Gamma_{kl}^i, S^{ik}, t_{ik}, T^{ik}, \lambda)$$

where:  $M^4$  is a 4-dim manifold

$\Gamma_{kl}^i$  is a symmetric connection

$S^{ik}, t_{ik}, T^{ik}$  are symmetric tensor fields on  $M^4$

$\lambda \geq 0$ , constant

- $t_{ik}$ : the time metric
- $s^{ik}$ : the space metric
- $\Gamma_{ik}^l$ : a connection, the gravitational field
- $T^{ik}$ : the matter tensor

The laws of the theory contain these fields and a real-valued parameters  $\lambda$  .

Here are some of the laws:

$$t_{ik}s^{kl} = -\lambda\delta_i^l .$$

$$t_{ik;l} = 0 , s^{ik}_{;l} = 0$$

$$R_{k\bullet m}^i = R_{m\bullet k}^i$$

$$R_{ik} = 8\pi(T_{ik}^{\bullet\bullet} - \frac{1}{2}t_{ik}T^j_{j\bullet})$$

$$T^{ik}_{;k} = 0$$

Matter model perfect fluid:

$$T^{ik} = (\rho + \lambda p)U^iU^k + ps^{ik}$$

where  $t_{ik}U^iU^k = 1, \rho > 0, \rho + \lambda p > 0$

The laws for these objects imply that for  $\lambda = 0$  we have Newton's theory and  $\lambda > 0$  is Einstein's theory.

For example: one can show that for  $\lambda = 0$  there exists a scalar  $t$ , absolute time, such that  $t_{ik} = t_{,i}t_{,k}$

A key object is the connection  $\Gamma_{kl}^i$  which in Newton's theory is defined by the motion of freely falling particles in the gravitational field.

note: Minkowski space:

$$\eta_{ik} = \text{diag}(-c^2, 1, 1, 1) , \quad \eta^{ik} = \text{diag}\left(-\frac{1}{c^2}, 1, 1, 1\right)$$

with  $\lambda = \frac{1}{c^2}$  we find

$$t_{ik} = -\lambda\eta_{ik} = \text{diag}(1, \lambda, \lambda, \lambda)$$

$$s^{ik} = \eta^{ik} = \text{diag}(-\lambda, 1, 1, 1)$$

meaningful for  $\lambda = 0$

If we use units, "c" appears naturally, its particular value depends on units.

A combined frame work for the theories. However, phenomena are described by solutions. Does the frame theory help to understand the relation between solutions in both theories?

## Newtonian Limit

The frame theory allows to define the Newtonian limit of a  $\lambda$  – family of solutions of Einstein’s theory:

$$g^{ik}(\mathbf{x}^j, \lambda), T^{ik}(\mathbf{x}^j, \lambda), 0 > \lambda$$

has a Newtonian limit, if the fields

$$g^{ik}, t_{ik} := -\lambda g_{ik}, \Gamma_{kl}^i, T^{ik}$$

have a limit for  $\lambda \rightarrow 0$  and this limit satisfies the assumptions of the frame theory for  $\lambda = 0$ .

example:

$$-\lambda^{-1} \left(1 - \frac{2\lambda}{r}\right) dt^2 + \frac{1}{1 - \frac{2\lambda}{r}} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

limits for  $g^{ik}, -\lambda g_{ik}$  .

Static fluid ball solutions have a Newtonian limit. More examples are known.

Conceptual problems concerning the meaning of  $c \rightarrow \infty$  can be clarified by considering the scalings of the fields which is implied by the freedom of choosing units.

How does one construct such families?

One makes an ansatz, ( as one does in derivations of p–N–expansions.)

For example: ( $\eta^{\text{ik}} = \text{diag}(-\lambda, 1, 1, 1)$ )

$$g^{\text{ik}} = \eta^{\text{ik}} + \lambda h^{\text{ik}}(\mathbf{x}^1, \lambda) \quad , \quad \mathbf{T}^{\text{ik}} = \dots$$

Putting this into the field equations gives a PDE system containing  $\lambda$  explicitly. However, if one does this just naively, there will be no PDE system for  $\lambda = 0$  because powers of  $\lambda, \lambda^{-1}$  will appear.

Ehlers found a way around this problem and proposed a set of variables such that the PDEs have a limit for  $\lambda = 0$ .

(never published first appeared in the PhD thesis of Lottermoser 1988)

Strategy:

1. find PDE system regular for  $\lambda = 0$
2. find  $\lambda$ –family of solutions of the constraint which have a Newtonian limit
3. Show that the evolved data have also a Newtonian limit

## Early Results

M. Lottermoser, PhD thesis 1988

A convergent post-Newtonian approximation for the constraint equations in general relativity

*Annales de H.P.* 57,n.3 (1992)

U.Heilig PhD thesis (1993)

On the Existence of Rotating Stars in General Relativity

*Commen.Math.Phys.*166(1995)

Both use the variables proposed by Ehlers:

$$\mathcal{G}^{\mathbf{ik}} := \frac{1}{\sqrt{|\det(\mathbf{g}^{\mathbf{ik}})|}} \mathbf{g}^{\mathbf{ik}}$$

$$\mathcal{U}^{\mathbf{ik}} := \frac{1}{4} \lambda^{-\frac{3}{2}} (\mathcal{G}^{\mathbf{ik}} - \mathcal{G}_0^{\mathbf{ik}})$$

$$\mathcal{G}_0^{\mathbf{ik}} = \text{diag}(-\lambda^{\frac{1}{2}}, \lambda^{-\frac{1}{2}}, \lambda^{-\frac{1}{2}}, \lambda^{-\frac{1}{2}})$$

(the density of the Minkowski metric  $\eta^{\mathbf{ik}} = \text{diag}(-\lambda, 1, 1, 1)$ )

This implies for the spacetime metric

$$g^{00} = -\lambda + \lambda^2 2U + \dots$$

$$g^{0I} = -\lambda^2 4W^I + \dots$$

$$g^{IK} = \delta^{IK} + \lambda 2U \delta^{IK} + \lambda^2 2 \left( (3U^2 - \text{Tr}Z \delta^{IK}) + 2Z^{IK} \right) + \dots$$

with  $U := \mathcal{U}^{00}$ ,  $W^J := \mathcal{U}^{0J}$ ,  $Z^{IK} := \mathcal{U}^{IK}$

The field equations in harmonic coordinates

$$(\bar{\mathcal{G}}_0^{\text{ik}} + 4\lambda^2 \mathcal{U}^{\text{ik}}) \mathcal{U}^{\text{jl}}{}_{,\text{ik}} + \mathbf{F}^{\text{jl}}(\lambda, \mathcal{U}, \partial \mathcal{U}) = 4\pi \mathbf{G} |\mathbf{d}| \mathbf{T}^{\text{jl}}$$

$$\bar{\mathcal{G}}_0^{\text{ik}} = \text{diag}(-\lambda, 1, 1, 1)$$

$$\mathbf{d} := \lambda \det(\mathcal{G}^{\text{ik}}) = 1 - \lambda 4U + \dots$$

This is a regular system also for  $\lambda = 0$ .

However, hyperbolic goes to elliptic!

$$\Delta U = 4\pi \mathbf{G} \rho$$

$$\Delta W^I = 4\pi \mathbf{G} J^I$$

$$\Delta Z^{\text{IJ}} = 4\pi \mathbf{G} S^{\text{IJ}} + U_{,\text{I}} U_{,\text{J}} + \frac{1}{2} |\text{grad} U|^2 \delta^{\text{IJ}}$$

$\rho, J^I, S^{\text{IJ}}$  are the matter fields at  $\lambda = 0$ .

Lottermoser studies solutions of the constraints:

$$4\pi G \lambda g T^{00} = \Delta U + F^{00} - \lambda Z^{CD}_{,CD} +$$

$$4\lambda^2 (Z^{CD} U_{,CD} - 2W^C_{,CD} W^D + U Z^{CD}_{,CD})$$

$$4\pi G \lambda g T^{0J} = \Delta W^J + F^{0J} - \lambda \dot{Z}^{JD}_{,D} + \lambda^2 Z^{CD} W^J_{,CD} \dots$$

Prescribe  $T^{00}, T^{0J}, Z^{IJ}, \dot{Z}^{IJ}$  (possible depending on  $\lambda$ .)

solve for  $U, W^J$

The harmonicity condition

$$\dot{U} = -W^j_{,j} \quad , \quad \dot{W}^K = -Z^{kL}_{,L}$$

determine the time derivatives of  $U, W^J$ .

For small  $\lambda$  asymptotically flat solutions do exist on  $R^3$ .

Note the "free data" for the gravitational field  $Z^{IJ}, \dot{Z}^{IJ}$ .

An important observation (Lottermose):

Suppose we can evolve these data and the solution has a limit, differentiable in  $\lambda$ . Then the time evolution equations imply for  $\lambda = 0$

$$\Delta Z^{IJ} - 4\pi G T_0^{IJ} + U_{,I} U_{,J} + \frac{1}{2} |\text{grad} U|^2 \delta^{IJ} = 0$$

This is a restriction on the datum which we considered as free!

Post-Newtonian expansion has to do with the differentiability in  $\lambda$

$$g^{ik} = \delta^{ik} + \lambda h_1^{ik}(x^1) + \lambda^2 h_2^{ik}(x^1) + \dots$$

On the formal level smoothness in  $\lambda$  implies that the "free gravitational data" are uniquely determined by the matter data!

Heilig used the equations above in a stationary situation. Then all equations are elliptic also for  $\lambda > 0$ .

With the implicit function theorem, he showed the existence of slowly rotating, axisymmetric fluid configurations. (for a restricted class of equations of state).

The family is analytic in  $\lambda$  and allows  $p$ - $N$ -expansions of any order.

Newtonian limit considerations are not just "philosophical"!

A.Rendall

The Newtonian limit for asymptotically flat solutions of the Vlasov–Einstein–system

*Commun.Math.Phys.* 163 (1994)

This is the first theorem, which shows the existence of time dependent  $\lambda$ -families with Newtonian limits.

The matter is described by kinetic theory. The key difficulty is to demonstrate, that the  $\lambda$ -family exist for an intervall  $0 < t < T$ , independent of  $\lambda$ .

## New Results

**T.Oliyныk**

**The Newtonian Limit for Perfect Fluids**

*Commen.Math.Phys. 276(2007)*

**Post–Newtonian expansion for perfect fluids**

*Commen.Math.Phys. to appear , (2008)*

**The fast Newtonian limit for perfect fluids**

*preprint (2009)*

**Cosmological post–Newtonian expansions to arbitrary order**

*preprint (2009)*

Oliynyk uses  $\bar{u}^{ik}$  a slight variation of the Ehlers variables

$$g^{ik} = \frac{\epsilon}{-\det(Q)} Q^{ik} \quad , \epsilon = \sqrt{\lambda}$$

$$Q^{00} = -\epsilon^2 + 4\epsilon^4 \bar{u}^{00}$$

$$Q^{0J} = 4\epsilon^3 \bar{u}^{0J}$$

$$Q^{IJ} = \delta^{IJ} + \epsilon^2 \bar{u}^{IJ}$$

The matter is a "Makino fluid", a fluid with a particular equation of state for which there is an existence theorem for finite bodies.

$$(p = K\rho^{(n+1)/n})$$

Matter variables: fluid flow  $w^J$ ,  $\alpha$  related to the density

Combine  $\mathbf{V} = (\bar{u}^{ik}, \partial_j \bar{u}^{ik}, w^j, \alpha)$

**Einstein–Euler in harmonic gauge**

$$\begin{aligned} \mathbf{b}^0(\epsilon \mathbf{V}) \partial_0 \mathbf{V} &= \frac{1}{\epsilon} \mathbf{c}^I \partial_I \mathbf{V} + \mathbf{b}^I(\epsilon, \mathbf{V})^I \partial_i \mathbf{V} + \\ &+ \mathbf{f}(\epsilon, \mathbf{V}) + \frac{1}{\epsilon} \mathbf{g}(\mathbf{V}) \mathbf{V} + \mathbf{h}(\epsilon) \end{aligned}$$

**a symmetric hyperbolic system;  $\mathbf{c}^I$  is constant!**

The term  $\frac{1}{\epsilon} \mathbf{g}(\mathbf{V}) \mathbf{V}$  can be moved to the term with  $\mathbf{c}^I$  by a non local transformation (Poisson integral) which changes  $\bar{u}^{00}$  to  $u^{00}$ . For  $\mathbf{W} = (u^{ik}, \partial_j u^{ik}, w^j, \alpha)$  one obtains

$$\begin{aligned} \mathbf{B}^0(\epsilon \mathbf{W}) \partial_0 \mathbf{W} &= \frac{1}{\epsilon} \mathbf{c}^I \partial_I \mathbf{W} + \mathbf{B}^I(\epsilon, \mathbf{W}) \partial_I \mathbf{W} + \\ &+ \mathbf{F}(\epsilon, \mathbf{V}) + \mathbf{F}(\epsilon, \mathbf{W}) + \mathbf{H}(\epsilon) \end{aligned}$$

**a non–local symmetric hyperbolic system.**

**Singular systems of this type are well studied.**

**Klainermann, Majda, Kreiss, Schochet**

The constraints for the initial data are treated as by Lottermoser:

Free data are:  $u^{IJ}, \partial_0 u^{IJ}, w^J, \alpha$

The remaining data  $u^{00}, u^{0K}$  are determined as solutions of the constraints (plus harmonicity).

Sophisticated functional analysis;  $\epsilon$  goes into the weights of some weighted Sobolev spaces.

Three cases are studied: (different "amount of radiation")

1.) Newtonian limit

$$u^{IJ}(0) = \epsilon z^{IJ}, \partial_0 u^{IJ}(0) = z_0^{IJ}, w_0^J, \alpha_0$$

The data determine solutions  $u_\epsilon^{IJ}, w_\epsilon^J, \alpha_\epsilon$  with a limit for  $\epsilon \rightarrow 0$  which satisfies the Newton – Euler–system. There are estimates of the type

$$\|\rho_\epsilon - \rho_0\| < \epsilon, \|\mathbf{u}_\epsilon^{ik} - \delta_0^i \delta_0^k \Phi\| < \epsilon$$

( $\Phi = u_0^{00}$ ) One has a limit in the sense of Ehlers after some time  $T_\star > 0$ .

## 2.) Fast Newtonian limit

Here the data are more general

$$\bar{\mathbf{u}}^{\text{IJ}}(\mathbf{0}) = \mathbf{z}^{\text{IJ}}, \quad \epsilon \partial_t \bar{\mathbf{u}}^{\text{IJ}}(\mathbf{0}) = \mathbf{z}_0^{\text{IJ}}, \quad \mathbf{w}_0^{\text{J}}, \quad \alpha_0$$

and convergence as described in 1) is again true after some time  $\mathbf{T}_\star > 0$ .

### 3.) Post–Newtonian expansion

$\bar{u}^{IJ}(0) = \epsilon z^{IJ}$ ,  $\partial_t \bar{u}^{IJ}(0) = z_0^{IJ}$  are determined from the matter data  $w_0^J$ ,  $\alpha_0$  by equations of the typ

$$\Delta Z^{IJ} - 4\pi G T_0^{IJ} + U_{,I} U_{,J} + \frac{1}{2} |\text{grad} U|^2 \delta^{IJ} = 0$$

It can be shown that the 1–p–N expansion exists

$$g_{00} = -\frac{1}{\epsilon^2} - 2\Phi_0 - \epsilon h_1^{00} - \epsilon^2 \left( 3(\Phi_0)^2 + h_2^{00} \right) + \mathcal{O}(\epsilon^3),$$

$$g_{0I} = \epsilon^2 h_1^{0I} + \epsilon^3 h_2^{0I} + \mathcal{O}(\epsilon^4)$$

$$g_{IJ} = \delta_{IJ} - 2\epsilon^2 \delta_{IJ} \Phi_0 - \epsilon^3 h_1^{IJ} - \epsilon^4 \left( (\Phi_0)^2 \delta_{IJ} + h_2^{IJ} \right) + \mathcal{O}(\epsilon^5).$$

With more assumption on the free data the second p–N expansion exists.

It is important to to note that higher order expansions in  $\epsilon$  can be generated for  $g_{ij}$ . However, these higher order terms will, in general, depend on  $\epsilon$  in a non-analytic fashion.

## Cosmological post-Newtonian expansion

Consider perfect fluid solutions of Einstein's theory on  $\mathbb{R} \times \mathbb{T}^3$

There are also Newtonian solutions "of this type".

The techniques used before can be applied and one obtains solutions with a Newtonian limit.

A surprising result is, that with appropriate restrictions on the initial data, one obtains p-N-expansions of any prescribed finite order!