The issue of time in generally covariant theories and Quantum Gravity

Rodolfo Gambini

With Jorge Pullin and Rafael Porto

INTRODUCTION

There is by now extensive literature addressing the problem of time in Classical and Quantum Gravity.

The heart of the problem lies in the fact that Einstein gravity is a fully constrained system whose *Hamiltonian* vanishes, thus *observable* quantities are those that commute with the constraints, e.g. Dirac Observables, and therefore they do not *evolve*.

I will discuss here two approaches to this problem.

Both have in common their relational character. In fact, one of the basic ingredients in the different proposals to describe evolution is the use of *relations* between different degrees of freedom in the theory .

The first approach is based on the concept of *relational or evolving Dirac observables.* Bergmann, DeWitt, Rovelli, Marolf.

There is a second type of approach that I would like to mention in this talk that is the conditional probabilities approach proposed by Page and Wootters.

I shall try to convince you that both approaches present problems and do not provide a completely satisfactory solution to the issue of the evolution. Problems are particularly acute when we try to compute propagators or assign probabilities to histories.

Here we are going to propose a marriage of these approaches that exploits the advantages of both.

OUTLINE

1) Evolving Dirac Observables in totally constrained systems.

2) Page Wootters conditional probabilities.

3) Conditional probabilities in terms of evolving Dirac observables

4) Real clocks and loss of unitarity

5) Open problems

1) Evolving Dirac Observables in totally constrained systems:

$$S = \int [p_a \dot{q}^a - \mu^\alpha \phi_\alpha(q,p)] d\tau$$

In the case of GR the constraints are first class

$$\phi_\alpha(q,p) = 0$$

$$\{\phi_\alpha(q,p), \phi_\beta(q,p)\} = C^\gamma_{\alpha\beta} \phi_\gamma(q,p)$$

$$H_T = \mu^\alpha \phi_\alpha(q,p)$$

The Hamiltonian vanishes: the generator of the evolution also generates gauge transformations

Dirac observables are gauge invariant quantities

$$\{O(q,p), \phi_\beta(q,p)\} \approx 0 \qquad \{O(q,p), H_T(q,p)\} \approx 0$$

Therefore, they are constants of the motion.

The issue of time: If the physically relevant quantities in totally constrained systems as general relativity are constants of the motion, how can we describe the evolution?

1) Gauge fixing:

$$\tau = f(q,p), \qquad \tau = q^0$$

2) Evolving observables: Bergmann, DeWitt, Rovelli, Marolf

$$\{Q_i(t),\phi_\alpha\} \approx 0 \qquad Q_i(t,q^a,p_a)\big|_{t=q^1} = q_i$$

For instance, for the relativistic particle $\phi = p_0^2 - p^2 - m^2$

Two independent observables:

$$p, X \equiv q - \frac{p}{\sqrt{p^2+m^2}} q^0, \qquad Q(t,q^a,p_a) = X + \frac{p}{\sqrt{p^2+m^2}} t$$

$$Q(t=q^0,q^a,p_a) = q$$

Notice that one needs to assume that there are variables as q^0 that are physically observable, even though they are not Dirac observables

The Quantum Evolving Observables.

Let us consider the elementary case of a non-relativistic free particle

$$S = \int d\tau [p_0 \dot{x}^0 + p\dot{x} - N(p_0 + \frac{p^2}{2m})] \qquad \mathcal{H}_{aux} = L^2(\mathrm{R}^2, dp_0 dp)$$

with self-adjoint Dirac observables $\{\hat{q} := x - \hat{x}^0 p/m, \hat{p}, \hat{X}(T) := \hat{q} + \hat{p}T/m\}$

$$X(T)|_{T=x^0} = x$$

And eigenvectors:

$$\psi_{p_1}(p, p_0) = \delta(p_0 + \frac{p^2}{2m})\delta(p - p_1)$$

$$\psi_{q_1}(p, p_0) = \delta(p_0 + \frac{p^2}{2m}) \exp ipq_1$$

$$\psi_{x_1,T}(p, p_0) = \delta(p_0 + \frac{p^2}{2m}) \exp i(px_1 - \frac{p^2}{2m}T).$$

which are solutions of the constraint.

It is now possible to introduce an inner product in the space of solutions of the constraint $f(p, p_0)\delta(p_0 + \frac{p^2}{2m})$ and define H_{phys}

$$< \psi_1|\psi_2 >_{phys} = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp_0 \delta(p_0 + \frac{p^2}{2m}) f^*{}_1(p, p_0) f_2(p, p_0) = \int_{-\infty}^{\infty} dp \tilde{f}_1^*(p) \tilde{f}_2(p)$$

Summarizing, the choice of clock variable $T = x^0$ leads to the standard form of the quantum free particle in the Heisenberg representation.
In particular, the transition amplitude is:

$$< \psi_{x,T} | \psi_{x',T'} >= [\frac{2\pi i(T-T')}{m}]^{-1/2} \exp \frac{im(x-x')^2}{2(T-T')}.$$

This choice of clock variable for the non-relativistic particle is unique, up to reparameterizations. Any other choice leads to evolving Dirac Observables that cannot be promoted to self-adjoint operators.

For instance if one takes the position as a clock variable: *T=x,* it leads to an evolving observable:

$$X^0(T) = (-q+T)\frac{m}{p} \qquad X^0(T)|_{T=x} = x^0$$

that is not self-adjoint due to the momentum in the denominator.

If the classical Dirac observable can be promoted to a self-adjoint operator in H_{phys}, one can show that there is an operator, *U(T)*, such that the evolution in the c-number parameter T is unitary. The requirement that the evolving observables be self-adjoint is very restrictive in any totally constrained system and imposes strong limitations on the type of clocks that can be used at the quantum level.

The issue of the parameter T

Evolving observables depend on a real parameter T. That is we are assuming that there is an external magnitud T, that is not represent by any quantum operator nor it belongs to any physical Hilbert space.
One may wonder about the meaning of the condition $q^0 = T$ in the generic situation in which the clock variable q^0 is not defined in H_{phys}

$$q^0 \,|\psi>_{ph} \notin H_{ph}$$

In the example of the free non relativistic particle this difficulty is not apparent because we simple consider T as the time measured by an external clock, but if one wants to seriously consider that it corresponds to some dynamical variable of the system q^0 one runs into troubles.
In any generally covariant system as general relativity the clock will be associated to certain physical sub-system with dynamical variables that will not be well defined in H_{phys}

Evolving constants are measurable quantities but, in the quantum realm, they depend on an external parameter, whose observation is not described by the theory.

Marolf in a very interesting paper gr-qc/0902.1551 has recently presented an implementation of the evolving Dirac observables. We consider however that the issue of the external parameter is still present in this implementation.

2) Conditional probabilities.

A second alternative that preceded the idea of evolving observables was a description of the evolution in terms of conditional probabilities.

Page and Wootters analyzed this issue many years ago. *Kuchař* noted that this procedure faces important difficulties, in particular it does not lead to the correct propagators. The problem is related again with the fact that the evolution variable cannot be defined in the physical space of states.

Recently the idea received attention by *Dolby* who proposed a new approach to the issue of the definition of conditional probabilities.

Hellmann, Mondragon Perez and Rovelli **Phys.Rev.D75:084033,2007** analyzed the issue of the definition of probabilities for sequences of measurements proposed by Dolby and concluded that they present interpretational problems, and that it is not clear to what measurement setup does these probabilities correspond.

They have also proposed a treatment for the description of a sequence of quantum measurements based only on single event probabilities.

3) Conditional probabilities in terms of evolving Dirac observables.

R.G. R. Porto J. Pullin and S. Torterollo: Phys.Rev.D79:041501R,2009.

As we have seen, both approaches require the use of variables which are not defined in the physical space.

This is the case of the clock variable in Page-Wootters or the variables identified with *T* in the evolving observable approach.

In this paper we will elaborate upon a different approach where all reference to external parameters is abolished, and evolving constants are used to define correlations between Dirac observables in the theory.

Thus, we shall assume that evolving constants are measurable quantities, provided they can be promoted to well defined self-adjoint operators in the physical Hilbert space, but we would fall short to introduce an intrinsic, and fully quantum relational description of evolution, unless we find a way to get rid of the dependence on external parameters.

From a physical point of view, any observable in the theory should be described by a quantum operator in the space of physical states, and that should be no different for time, which it is ultimately measured by physical clocks obeying the laws of quantum mechanics.

First you choose an evolving observable as your clock, let us call it *T(t)* .
Then one identifies the set of observables $O_1(t),...O_N(t)$ that commute with T and describe the physical system whose evolution one wants to study and compute

$$\mathcal{P}\left(O \in [O_0 - \Delta O, O_0 + \Delta O] | T \in [T_0 - \Delta T, T_0 + \Delta T]\right) = \lim_{\tau \to \infty} \frac{\int_{-\tau}^{\tau} dt \, \mathrm{Tr}\left(P_{O_0}(t) P_{T_0}(t) \rho P_{T_0}(t)\right)}{\int_{-\tau}^{\tau} dt \, \mathrm{Tr}\left(P_{T_0}(t)\rho\right)}$$

Notice that I have changed the notation and the external parameter is now called *t.*

In other words, *t* is the parameter associated to the variable used to define the evolving observables. This variable is treated as an ideal unobservable quantity.

Notice that we are integrating in the "ideal" time *t*.
The experimental setup we have in mind is to consider an ensamble of non-interacting systems with two quantum variables each to be measured. Each systems is equipped with a recording device that takes a single snapshot of O and T at a "random" unknown value of the "ideal" time *t*. One takes a large number of such systems, launches them all in the same quantum state, "waits for a long time" and conclude the experiment. One computes how many times $n(T_j, O_j)$ each reading with a given value $T = T_j, O = O_j$ occurs.
From here one can immediately compute a joint probability in the limit of infinite systems, $P(O_j, T_j)$
We have assumed by simplicity magnitudes with discrete spectra.

A simple example.

One considers the constrained system:

$$\phi = p_0 + H(q^a, p_a) = 0$$

with $H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m}$ we have two free particles and one can define:

$$X_1(t) = q^1 - \frac{p_1}{m} q^0 + \frac{p_1}{m} t$$

$$X_2(t) = q^2 - \frac{p_2}{m} q^0 + \frac{p_2}{m} t$$

$$X_1(t)|_{t=q^0} = q^1$$

I am using q^0 as unobservable parameter and compute $P(X_2 \mid X_1)$

We can then write the conditional probabilities that yield the propagators,

$$P(X_2^f \,|\, T_2 = X_1^f, X_2^i, T_1 = X_1^i, \rho) \equiv$$

$$\lim_{\tau\to\infty} \frac{\int_{-\tau}^{\tau} dt \int_{-\tau}^{\tau} dt' Tr(P_{X_2^f, X_1^f}(t) P_{X_2^i, X_1^i}(t') \rho P_{X_2^i, X_1^i}(t'))}{\int_{-\tau}^{\tau} dt \int_{-\tau}^{\tau} dt' Tr(P_{X_1^f}(t) P_{X_2^i, X_1^i}(t') \rho P_{X_2^i, X_1^i}(t'))}$$

This expression yields the propagator for the system to move from

X_2^i, X_1^i to X_2^f, X_1^f

Notice that in particular no assumption about the relative ordering of the unobservable variables *t* and *t'* is needed.

One can show that this expression yields the correct propagator. In the example of the previous slide:

$$P\left(x_2'|x_1', x_2, x_1, \rho_0\right) \sim$$
$$\lim_{\tau\to\infty} \int_0^\tau dt' |\langle x_2', t'|x_2, t(x_1)\rangle|^2 \mathcal{P}_{x_1'}(t') \Delta x_2$$

Everything is given in terms of the Dirac Observables $X_1(t), X_2(t)$

4) Real clocks and loss of unitarity.

Let us come back to the previous result

$$P\left(x_2'|x_1', x_2, x_1, \rho_0\right) \sim \lim_{\tau\to\infty}\int_0^\tau dt' |\langle x_2', t'|x_2, t(x_1)\rangle|^2 \mathcal{P}_{x_1'}(t')\Delta x_2$$

$$\int \mathcal{P}_{x'_1}(t')dt' = 1$$

And $\mathcal{P}_{x'_1}(t')$ can be interpreted as the probability that the external unobservable time q^0 is t' when the variable taken as a clock reads x'_1

This probability will be controlled by the position of the peak and the width of the wave packet of the particle 1. If $\mathcal{P}_{x'_1}(t')$ were a Dirac delta we would recover the exact ordinary non-relativistic propagator.

The use of real clocks may lead to a loss of quantum coherence and therefore to corrections to the standard propagator.

$$P(x'_2 | x'_1, x_2, x_1) = \int Tr[\rho^H_{x_2x_1} P^H_{x'_2}(t')]\mathcal{P}_{x'_1}(t')dt' = \int Tr[\rho^S_{x_2x_1}(t')P^S_{x'_2}]\mathcal{P}_{x'_1}(t')dt' = Tr[\rho_{x_2x_1}(x'_1)P_{x'_2}]$$

$$\rho(T = x'_1) = \int dt' \mathcal{P}_{x'_1}(t')U(t', t(x_1))\rho_{x_2x_1}U^+(t', t(x_1))$$

We have therefore ended with the standard probability expression with an "effective" density matrix in the Schrödinger picture given by $\rho(T)$
Unitarity may be lost since one ends up with a density matrix that is a superposition of density matrices associated with different values of *t*

The origin of the lack of unitarity is the fact that definite statistical predictions are only possible by repeating an experiment. If one uses a real clock, which has quantum fluctuations, each experimental run will correspond to a different value of the evolution parameter. The statistical prediction will therefore correspond to an average over several intervals, and therefore its evolution cannot be unitary.

The underlying unitary evolution of the evolving constants in the ideal time *t* is crucial, yet unobservable. All we observe are the correlations in physical time, then it is not surprising those present a fundamental level of decoherence due to the intrinsically quantum and gravitational limitations of real clocks.

If we assume the “real clock" is behaving semi-classically.

$$\mathcal{P}_t(T) = f(T-t) = \delta(T-t) + a(T)\delta'(T-t) + b(T)\delta''(T-t) + \ldots$$

The Schrödinger evolution is modified: RG, R. Porto, JP, NJP 6, 45 (2004)

$$-i\hbar\frac{\partial\rho}{\partial T} = [\hat{H},\rho] + \sigma(T)[\hat{H},[\hat{H},\rho]] + \ldots$$

Where *σ(T)* is the rate of spread of the wave function of the clock: $\sigma(T) = \partial b(T)/\partial T$.
What are the consequences of the extra term? If we assume *σ* is constant, the equation can be solved exactly and one gets that the density matrix in an energy eigen-basis evolves as

$$\rho_{nm}(t) = \rho_{nm}(0)e^{-i\omega_{nm}t}e^{(-\sigma(\omega_{nm})^2)t}$$

Where the omega’s are the Bohr frequencies associated with the eigenvalues of H. $\omega_{mn} = E_m - E_n$

Therefore, the off-diagonal elements of the density matrix decay to zero exponentially, and pure states generically evolve into mixed states. Quantum mechanics with real clocks therefore does not have a unitary evolution.

Limitations to how good a clock or a rod can be

The effects are more pronounced the worse the clock is. Which raises the question: is there a fundamental limitation to how good a clock can be?

There are many phenomenological arguments based on quantum and gravitational considerations that lead to estimates of such a limitation,
(Salecker-Wigner and Ng, Karolyhazy, Lloyd, Hogan, Amelino Camelia) $\delta T = T^{1/3} t_p^{2/3}$

We will not enter into the analysis of these phenomenological estimations, but it is important to remark that the evolution with real clocks will not be unitary if the spread in the error of the clock grows with time with some power of *T*.

That is, if $\delta T = t_p$ the evolution is unitary, but if $\delta T = T^a t_p^{1-a}$ $a > 0$ there would exist a fundamental loss of unitarity.

In other words, the here proposed description of the evolution is perfectly unitary if clocks are only limited by the Planck time.

Conclusions:

- Using evolving constants of the motion in the conditional probability interpretation of Page and Wootters allows to correctly compute the propagator and assign probabilities to histories.
- The resulting description is entirely in terms of Dirac observables.
- There are corrections to the propagator due to the use of “real clocks and rods” to measure space and time.

Happy Birthday Abhay!