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ISOLATED HORIZONS AND BLACK HOLE MECHANICS

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Abstract

The zeroth and first laws of black hole mechanics are traditionally formulated in terms of a class of stationary space-times containing event horizons. However, this class of solutions is too restrictive to include a variety of physically interesting situations. This thesis describes the extension of the zeroth and first laws to a much broader class of space-times containing isolated horizons. A space-time representing a black hole which is itself in equilibrium, but whose exterior contains radiation, admits such a horizon. Using Hamiltonian techniques, quasi-local definitions of the "extrinsic parameters" of a black hole — the quantities which are related by the first law — are formulated for generic isolated horizons. These definitions reveal a remarkable connection between the first law and the classical Hamiltonian formalism.

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Chapter 1

Introduction

The analogy between the laws of black hole mechanics and those of ordinary thermodynamics is one of the most remarkable results to emerge from classical general relativity [1, 2, 3, 4, 5]. However, the traditional framework for these results depends critically on certain structures which are not present in most cases of physical interest. This thesis presents a different framework for the laws of black hole mechanics which is adapted to a much broader class of situations.

The zeroth and first laws of thermodynamics apply to equilibrium situations and small departures therefrom. By analogy, one expects the zeroth and first laws of black hole mechanics should also apply to equilibrium configurations, i.e., to black holes which are isolated from infalling matter and radiation. In standard treatments, isolated black holes are generally represented by *stationary* solutions to the field equations which admit a time-translational Killing fields *everywhere* in space-time, not just in a neighborhood of the black hole itself. While this simple idealization is a natural starting point, it seems overly restrictive from a physical point of view. In particular, the global requirement of stationarity does not admit radiation, gravitational or otherwise, anywhere in space-time. Physically, it should be sufficient to impose *local* conditions which ensure *only that the black hole itself is isolated*. That is, it should suffice to demand only that the intrinsic geometry of the black hole be "time-independent," whereas the geometry outside may be dynamical and admit gravitational and other radiation. Indeed, the traditional viewpoint in ordinary thermodynamics is very similar: when considering a classical gas, for example, one only assumes the system under consideration is in equilibrium, not the entire universe. The prototypical example of a black hole in equilibrium is that of the final stages of a gravitational collapse where the black hole has already formed and "settled down." In such situations, there is likely to be gravitational radiation and non-stationary matter far away from the black hole, whence the space-time as a whole is not expected to be stationary. Nevertheless, one expects black hole mechanics should incorporate such situations.

A second limitation of the standard framework for black hole mechanics lies in its dependence on *event horizons*. Physically, these surfaces represent causal boundaries in space-time through which light cannot escape to infinity. However, by their very definition, event horizons can only be constructed retroactively, after knowing the *complete* evolution of space-time. The difficulty with this picture is brought out by the example of figure 1.1 wherein a star of mass M initially undergoes a gravitational collapse and settles down to an equilibrium state. The null surface Δ_1 is like a causal boundary in space-time in the sense that it is foliated by a family of marginally trapped 2-surfaces[1]. If nothing further happens, Δ_1 would be part of the event horizon of the black hole. But suppose instead, after a very long time, a matter shell with mass δM collapses into the black hole. The black hole will quickly settle down to a new equilibrium state described by the surface Δ_2 and, assuming nothing further happens, Δ_2 *will* be part of the event horizon. However, if one continues the event horizon backward, one will find it actually lies slightly outside the surface Δ_1. Thus, although the equilibrium state Δ_1 may persist for a very long time (even by astro-physical standards), it cannot be treated in conventional formulations of black hole mechanics. On physical grounds, this exclusion seems unreasonable. Surely one should be able to establish the standard laws of mechanics not only for the event horizon but also for Δ_1.

The key ingredient in the framework for black hole mechanics presented here is the notion of an *isolated horizon* [6, 7, 8, 9, 10, 11]. An isolated horizon is defined as a null surface in space-time at which certain boundary conditions hold. Most importantly, any isolated horizon is foliated by a family of marginally trapped surfaces. An event horizon is always an isolated horizon, but the later is a much more general concept. The key point is that the definition of an isolated horizon is made *locally*, without reference to infinity, and constrains only the geometry of the horizon itself. In particular, the surface Δ_1 (as well as Δ_2) in figure 1.1 is an isolated horizon. A number of other physically important examples of isolated horizons will be given below.

In addition to overcoming the two limitations described above, the isolated horizon framework provides a natural point of departure for quantization and entropy calculations [12, 13]. In contrast, standard treatments of black hole mechanics are often based on differential geometric identities and are not well-suited to quantization. The existence of

[1] Physically, a marginally trapped surface is a space-like 2-surface with the property that an outward-bound light-front starting on the surface will not expand. In other words, light which travels radially outward from one of the marginally trapped leaves of Δ_1 actually propagates along the surface Δ_1 itself. Any ray originating on Δ_1 which does not travel radially outward propagates to the interior of Δ_1 in space-time.

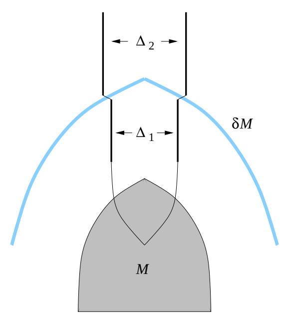

Figure 1.1: A spherical star of mass M undergoes collapse. Later, a spherical shell of mass δM falls into the resulting black hole. While Δ_1 and Δ_2 are both isolated horizons, only Δ_2 is part of the event horizon.

exact, non-perturbative, quantum calculations of the statistical entropy of isolated horizons provides direct evidence of the physics underlying the analogy between black hole mechanics and thermodynamics. In traditional treatments, this analogy is primarily supported by Hawking's celebrated, semi-classical result [5] on black hole radiance.

The above arguments illustrate how isolated horizons model a wider variety of black holes in equilibrium than the standard stationary examples. At first, it may appear that only a small extension of the standard framework is needed to formulate the laws of black hole mechanics for more general isolated horizons. However, this is not the case. Without access to globally defined Killing fields, even the definitions of many of the black hole's "extrinsic parameters" (mass, angular momentum, surface gravity, rotational velocity, etc.) present significant challenges. For example, in the stationary context, one identifies black hole mass with the ADM mass defined at spatial infinity. In the presence of radiation, this simple strategy is no longer viable since radiation fields far from the black hole also contribute to the ADM mass. To formulate the first law, a new definition of black hole mass is needed. As a second example, consider the definition of surface gravity. In the stationary context, every event horizon is generated by a Killing field which becomes null on the horizon surface; the surface gravity is defined as the acceleration of that Killing field at the horizon. Now, even if space-time admits such a horizon-generating Killing field in a neighborhood of the horizon — already a stronger condition than that contemplated here — the notion of surface gravity is ambiguous since constant rescalings of the Killing field change its acceleration. When a *global* Killing field exists, the ambiguity is removed by normalizing the Killing field through its properties at *infinity*. Thus, contrary to intuitive expectation, the standard notion of surface gravity for a stationary black hole refers not just to the structure at the horizon, but also to infinity. Similar problems occur with the definitions of angular momentum, rotational velocity and, in the Einstein–Maxwell case, electric potential. All of these problems must be overcome for the laws of black hole mechanics even to be formulated, much less proved. Fortunately, the isolated horizon boundary conditions allow a phase space formulation of an isolated black hole and this formulation offers a great deal of guidance in making the appropriate definitions. Apart from the conceptual problems mentioned here, a host of technical issues must also be resolved in the phase space construction. In Einstein–Maxwell theory, for example, the space of stationary Kerr–Newman black hole solutions is finite-dimensional whereas the space of solutions admitting isolated horizons is *infinite*-dimensional. As a result, the introduction of a well-defined action principle is subtle and the Hamiltonian framework acquires qualitatively new features.

This thesis is organized as follows. Chapter 2 recalls the traditional framework for black hole mechanics. In particular, the notion of a Killing horizon, its relation to the zeroth law, and the derivation of the first law in the stationary context are reviewed. The second law is not discussed since it applies to dynamical, not equilibrium, situations and therefore lies outside the scope of this thesis.

Chapter 3 introduces the boundary conditions which define a *isolated horizon* and discusses a number of examples. The primary focus in Chapter 3 is on Einstein–Maxwell theory, although some of the structure with more general matter fields is also explored. The consequences of these boundary conditions which are relevant to the discussion of black hole mechanics are derived in Chapter 4. In particular, the zeroth law of black hole mechanics for general isolated horizons is established. The last part of Chapter 4 demonstrates the existence of a preferred foliation of the horizon which will be useful in the following chapters.

Using the previous results, Chapter 5 formulates an action principle describing an asymptotically flat space-time with an interior boundary at a single isolated horizon. It uses a first-order action for general relativity in terms of a tetrad and a (real) Lorentz connection. The action gives rise to a (covariant) phase space for the theory on which certain Hamiltonian generating functions are constructed. Chapter 6 is devoted to the derivation of the first law of black hole mechanics for isolated horizons in vacuum general relativity. Angular momentum is defined for a class of horizons which admit an axial symmetry. The energy of horizons in this same class is then defined, leading naturally to a generalized form of the first law of black hole mechanics. The last part of Chapter 6 examines the structure of the first law for isolated horizons more closely, revealing an intimate connection with the phase space formalism. It also discusses the issue of picking the rest frame of an isolated horizon, thereby defining its mass. The inclusion of the usual stationary solutions in the isolated horizon formalism is also discussed here.

Chapter 7 contains a review of the results in the thesis. Appendix A extends the results of Chapter 6 to include the electromagnetic field. Finally, since it is useful at a number of points throughout our discussion, the Newman–Penrose formalism is reviewed in Appendix B.

Throughout this thesis, tensor indices are supressed wherever doing so does not obscure the results. Where space-time indices do appear, they are denoted by lower-case latin letters $(a, b, c, \ldots)$. Internal frame indices are denoted by upper-case latin letters $(I, J, K, \ldots)$. In tensorial expressions where indices are supressed, the metric is used freely to convert covariant tensors to contravariant and vice versa. The expression $V \lrcorner\, \omega$ denotes the contraction of the vector field V into the first index of the differential n-form ω. Finally, for symmetric, covariant tensors of valence two such as the Ricci tensor R, the contraction of a vector field V on one of its indices is denoted $R(V)$.

Chapter 2

Classical Black Hole Mechanics

In this chapter, we review the traditional framework for black hole mechanics. We recall Hawking's area theorem [14] and Bekenstein's realization [1, 2] of its analogy with the second law of black hole mechanics. We then examine the proofs of the zeroth and first laws of black hole mechanics using (a) the black hole uniqueness theorems and (b) the structure of Killing horizons. We conclude this chapter with a brief critique of the traditional framework. There exist a number of excellent reviews of this material in the literature [15, 16, 17]; the presentation here is mainly for completeness.

2.1 EVENT HORIZONS AND THE AREA LAW

From a physical point of view, a black hole is a region of space-time from which gravitational attraction prevents even light escaping. In asymptotically flat space-times, this notion is represented mathematically as a region lying outside the causal past of future null infinity [18, 15]. The boundary of such a region is known as the black hole's *event horizon* $\mathscr{H}$. Note that, by definition, the event horizon is a global construction; the entire space-time history must be known before it can be found.

The event horizon $\mathscr{H}$ is a null hyper-surface in space-time which is generated by a family of future-inextensible null geodesics without caustics [18, 19]. That is, the expansion of the null congruence generating $\mathscr{H}$ cannot become negative infinity anywhere on $\mathscr{H}$. This is the essential fact in proving Hawking's area theorem [14, 19]: the area of a black hole's event horizon can never decrease with time. Under the simplifying assumption that the generators of the horizon are geodetically complete, a simple proof of this proposition can be made as follows. The Raychaudhuri equation (see section 4.1) for an affinely parameterized null geodesic congruence reads

$$-\left[\frac{d\theta}{d\lambda}+\frac{1}{2}\theta^2\right]=\sigma_{ab}\sigma^{ab}-\omega_{ab}\omega^{ab}+R_{ab}\xi^a\xi^b, \tag{2.1}$$

where λ is the affine parameter along the generator, ξ^a, of the congruence, R_{ab} is the Ricci tensor and θ, σ_{ab} and ω_{ab} denote the expansion, shear and twist of ξ^a, respectively. In the case where ξ^a generates the event horizon, its twist automatically vanishes. Furthermore, if one assumes the Einstein equations hold with matter satisfying the null energy condition ($T_{ab}k^a k^b \geq 0$ for all null k^a), the right side of this equation is non-negative. Thus, 2.1 yields a differential inequality which can be integrated to show

$$\frac{1}{\theta(\lambda)} \geq \frac{1}{\theta(\lambda_0)} + \frac{\lambda - \lambda_0}{2}. \tag{2.2}$$

It follows that, if $\theta(\lambda_0)$ is negative anywhere on the event horizon, then $\theta(\lambda_\star)$ will tend to negative infinity for some $\lambda_\star \leq \lambda_0 + 2[-\theta(\lambda_0)]^{-1}$. Thus, the expansion of the horizon must be everywhere non-negative and it follows that the horizon area can never decrease with time. In the case where ξ^a is *not* geodetically complete, this argument breaks down since the parameter $\lambda_\star$ may never be reached. In this case, however, one can again show the expansion θ is always non-negative under the technical assumption that the space-time in question is "strongly asymptotically predictable" [19, 18]. The details of this argument are not central to the discussion here.

Hawking's area law is remarkably similar to the second law of thermodynamics in that each states a certain quantity cannot decrease in time. In thermodynamics that quantity is the entropy, in general relativity it is the area of a black hole's event horizon. At first, the similarity may appear coincidental since the two laws arise in very different contexts: the area law is a geometric theorem whereas the second law of thermodynamics arises from statistical considerations. Nonetheless, motivated by the area law [14] as well as by earlier work of Floyd and Penrose [20], Christodoulou and Ruffini [21] and Carter [22], Bekenstein was led to propose [1] that the analogy should, in fact, be taken seriously and the statistical entropy of a black hole identified with (some multiple of) its horizon area. Bekenstein defended this proposition by noting that matter falling through the event horizon could carry with it some statistical entropy, thereby decreasing the total entropy of the accessible universe and violating the second law of thermodynamics. On the other hand, he reasoned, the infalling matter would expand the area of the event horizon. By considering the example of a small, spherical, semi-classical particle falling into a black hole, Bekenstein showed the entropy of the black hole should increase by the ratio of the increase in its horizon area to the square of the Planck length times a dimensionless constant of order unity. Consequently, the statistical entropy of a black hole should be proportional to its area over the square of the Planck length. Note that the appearance of Planck's constant implies the statistical entropy of a black hole *must* be quantum mechanical in origin.

2.2 THE MECHANICS OF THE KERR–NEWMAN BLACK HOLE

If the area of a black hole measures its entropy, it is natural to ask which observables measure its other thermodynamic properties. Bekenstein [1] analyzed this issue using formulas for the Kerr–Newman class of charged, rotating black holes. He discovered there is a simple analog of temperature for a black hole and derived two laws of black hole mechanics analogous to the zeroth and first laws of thermodynamics. We review his construction here.

Before deriving the black hole results, let us recall the zeroth and first laws of thermodynamics. The zeroth law simply states the temperature T of a body in thermal equilibrium is uniform throughout the body. Note this law applies only to non-dynamical, equilibrium states of a thermodynamic system. The first law, in contrast, applies to motions in the space of equilibrium states of a system, known as the thermodynamic state space. It takes the form of the differential identity

$$\mathrm{d}E = T\mathrm{d}S + W, \tag{2.3}$$

where E is the energy of the system, T is its temperature, S is its entropy and W is a 1-form on state space representing the work done on the system by external agents in the course of changing its thermal state. Generally, W has the form $W = \sum F_i \,\mathrm{d}\alpha^i$, where α^i are certain quantities describing the state of the system and F_i are the applied external "forces." The reason thermodynamics is such a useful tool is that it reduces the study of large, complicated physical systems to a question of the behavior of a handful of macroscopic parameters. Experimentally, one is seldom interested in the detailed internal dynamics of the system, but rather in the behavior of its measurable bulk parameters. These quantities, known as the *extrinsic parameters* of the system, are exactly those which appear in the first law. For example, the extrinsic parameters of an ideal gas include its internal energy, temperature and entropy as well is its volume ($\alpha^1 = V$) and pressure ($F_1 = -P$). Moreover, the first law implies only a subset of the extrinsic parameters can be taken as independent coordinates on state space. For an ideal gas, specifying any pair of (E, T, S, P, V) determines its thermal state. One is, however, free to pick any pair of independent variables one likes. The values of the remaining extrinsic parameters can be determined in terms of the independent variables using the first law.

In the case of black hole mechanics the critical point is that the geometry of a Kerr–Newman black hole is completely determined by only three independent parameters. The metric for the Kerr–Newman solution is (in the standard Boyer–Lindquist coordinates t, r, θ and ϕ)

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta - a^2\sin^2\theta}{\Sigma}\,\mathrm{d}t^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\phi + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 \\ &+ \Sigma\,\mathrm{d}\theta^2 + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\,\sin^2\theta\,\mathrm{d}\phi^2 \end{aligned} \tag{2.4}$$

and its Maxwell potential is

$$\mathbb{A} = -\frac{Qr}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\right), \tag{2.5}$$

where $a := J/M$ is the angular momentum per unit mass of the black hole, Q is its electric charge, and the functions Δ and Σ are given by

$$\Delta := r^2 + a^2 + GQ^2 - 2GMr \quad \text{and} \quad \Sigma := r^2 + a^2\cos^2\theta. \tag{2.6}$$

All of these solutions are stationary and axi-symmetric under the Killing fields ∂_t and ∂_ϕ, respectively. The function Δ has two zeros at $r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$, with the event horizon located at $r = r_+$. Using the metric 2.4, one easily finds the area of the horizon is $A = 4\pi(r_+^2 + a^2)$. Furthermore, the Killing field

$$\xi := \partial_t + \Omega\partial_\phi \quad \text{with} \quad \Omega = \frac{a}{r_+^2 + a^2} \tag{2.7}$$

is, at the horizon, tangent to its null geodesic generators. The quantity Ω is physically interpreted as the *rotational velocity* of the black hole. In general, the vector field ξ, though geodetic when restricted to the horizon, is not affinely parameterized and its acceleration κ defines the *surface gravity* of the black hole. Using the metric 2.4, it is straightforward to calculate

$$\nabla_\xi\,\xi =: \kappa\xi \quad \text{with} \quad \kappa = \frac{r_+ - GM}{r_+^2 + a^2}. \tag{2.8}$$

For the Maxwell field, note that the gauge of the vector potential is chosen such that it falls off to zero at infinity. Since the solution is stationary, the *electric potential* Φ of the horizon is well-defined and its value is

$$\Phi := -\xi \lrcorner\, \mathbb{A} = \frac{Qr_+}{r_+^2 + a^2}. \tag{2.9}$$

These quantities (area, rotational velocity, surface gravity and electric potential), together with the three independent parameters (mass, angular momentum and electric charge) of the Kerr–Newman solutions, form the complete set of "extrinsic parameters" for a black hole in Einstein–Maxwell theory. These are the parameters to which the laws of black hole mechanics refer.

The derivation of the first law of of black hole mechanics in Bekenstein's framework is a simple exercise in differential calculus. The independent parameters of the black hole in the calculations above are taken to be its mass M, angular momentum ratio $a = J/M$ and electric charge Q. However, for our purposes it will be useful to replace the mass with the geometric radius R of the horizon (i.e., $A = 4\pi R^2$) and a with the angular momentum itself in this list. The remaining "extrinsic parameters" can be written in terms of R, J and Q as

$$\begin{aligned} M &= \frac{\sqrt{(R^2+GQ^2)^2+4G^2J^2}}{2GR} & \Omega &= \frac{2GJ}{R\sqrt{(R^2+GQ^2)^2+4G^2J^2}} \\ \kappa &= \frac{R^4-G^2(Q^4+4J^2)}{2R^3\sqrt{(R^2+GQ^2)^2+4G^2J^2}} & \Phi &= \frac{Q}{R}\,\frac{R^2+GQ^2}{\sqrt{(R^2+GQ^2)^2+4G^2J^2}}, \end{aligned} \tag{2.10}$$

The mass M is a natural measure of the energy in a black hole system. Since the first law of thermodynamics involves the gradient of the energy function, we seek an analogous relation by varying the mass M within the class of Kerr–Newman black holes. It is straightforward to calculate

$$\delta M = \frac{\kappa}{8\pi G}\,\delta A + \Omega\,\delta J + \Phi\,\delta Q. \tag{2.11}$$

This is the *first law of black hole mechanics* for electrically charged, rotating black holes. We have already seen the entropy of a black hole is proportional to its area, so the analogy between 2.11 and 2.3 suggests identifying its temperature with (some multiple of) the surface gravity. Using the explicit expression 2.10, we immediately find the *zeroth law of black hole mechanics*: the surface gravity is constant over the horizon of a black hole in equilibrium. Again, this is directly analogous to the zeroth law of thermodynamics.

Despite the two analogies, however, there is an apparent problem with the identification of surface gravity with temperature for black holes. Since a classical black hole is a perfect absorber, its thermodynamic temperature should always be zero and the surface gravity clearly is not. To resolve this problem, recall Bekenstein's calculations indicated the statistical entropy of the black hole is inversely proportional to Planck's constant, so by 2.11 its temperature should actually be identified with some multiple of Planck's constant times the surface gravity. Thus, the black hole's temperature *is* always zero in the classical limit where $\hbar \to 0$. Hawking's famous result on black hole radiance [5] provides additional support for this identification in the semi-classical regime. He finds a quantum field in a black hole space-time will, to a static observer at infinity, appear to be in a thermal state with temperature $T = \hbar\kappa/2\pi$. This explains why the results we have found here are often called the laws of black hole *mechanics*, rather than of black hole *thermodynamics* (though this second term is also used quite frequently). They are purely classical, geometric results which have nothing to do with a statistical analysis of black hole systems. It is a remarkable, and still largely mysterious, fact that the statistical properties of black holes in *quantum* gravity seem to be reflected so clearly within *classical* general relativity. This property of the laws of black hole mechanics is primarily responsible for the great deal of discussion they have provoked over the past three decades.

2.3 KILLING HORIZONS AND BLACK HOLE MECHANICS

In light of the black hole uniqueness theorems [23, 24] which assert the only stationary, axi-symmetric solutions of Einstein–Maxwell theory with regular event horizons are the Kerr-Newman black holes, Bekenstein's calculations are entirely satisfactory to establish the laws of black hole mechanics in this context. However, at the time Bekenstein did this work, those uniqueness theorems were not yet proved and, moreover, his results do not allow more general matter fields. Bardeen, Carter and Hawking derived the same results [3] using a different technique which applies to more general stationary black hole space-times. It will be useful to review their constructions here.

Let us begin by recalling a number of definitions. First, a *stationary* space-time is one with a one-parameter group of isometries generated by a Killing field t which becomes unit and time-like at infinity. Second, a space-time with a stationary Killing field t which, in addition, is hypersurface-orthogonal everywhere is said to be *static*. Third, a space-time is *axi-symmetric* if it admits a one-parameter group of isometries which, at infinity, correspond to rigid rotations. Fourth, a stationary, axi-symmetric space-time is *circular* if the 2-planes orthogonal to both t and the axial Killing field ϕ are everywhere integrable[1] (i.e., if the space-time is foliated by 2-dimensional surfaces orthogonal to both t and ϕ). Finally, a *Killing horizon* is a null surface $\mathscr{K}$ whose generators coincide with the orbits of a one-parameter group of isometries. Equivalently, there exists a Killing field which is normal to $\mathscr{K}$, though it needn't be null or hypersurface-orthogonal anywhere else in space-time.

The Kerr–Newman solutions, in particular, are circular and their event horizons are Killing horizons under the isometries along ξ defined by 2.7. This situation is actually much more common than one might expect at first. The rigidity theorems of Hawking [19] and

[1] This property is also known as the "t-ϕ orthogonality property" in the literature [17].

Carter [4] imply the event horizons of stationary black holes are generically Killing horizons. Hawking's proof proceeds by assuming the Einstein–Maxwell equations are satisfied with no other matter present in space-time and shows the event horizon of any stationary, but non-static, black hole must be generated by a second, linearly independent Killing vector field. Additionally, Hawking's results indicate that stationary black hole solutions in Einstein–Maxwell theory must also be axi-symmetric (though not necessarily circular). Carter's proof, on the other hand, does not make use of the field equations for general relativity but applies only to static or circular black hole space-times. His results indicate the event horizon is generated by a unique combination of the two Killing fields,

$$\xi = t + \Omega\phi, \tag{2.12}$$

with Ω constant over the horizon. As before, Ω is interpreted as the angular velocity of the horizon. The two proofs complement one another. Hawking's theorem applies to general stationary black holes, but uses the Einstein–Maxwell equations, while Carter's uses no field equations but assumes circularity. However, the physical content of their results is unambiguous: physically interesting stationary black holes have Killing horizons. This is a particularly important result since the proofs of the laws of black hole mechanics we are about to examine depend on the Killing horizon structure.

Consider a Killing horizon $\mathscr{K}$ generated by a Killing field ξ. The surface gravity of $\mathscr{K}$ is defined in terms of ξ just as in 2.8. Since ξ is a Killing vector, this definition can be rewritten in a number of equivalent forms, but one which is particularly useful here is

$$\nabla_\xi \xi = \kappa\xi \quad \Leftrightarrow \quad \mathrm{d}(\xi\cdot\xi) = -2\kappa\xi. \tag{2.13}$$

It follows immediately from the second expression here that the surface gravity is always constant along each generator of a Killing horizon. To prove the zeroth law of black hole mechanics, one must show it is also constant from one generator to another. Bardeen, Carter and Hawking did this by assuming the Einstein equations hold with general matter satisfying the *dominant energy condition* which asserts that the vector $-T(k)$ is causal (i.e., future-directed and either time-like or null) for any causal vector k. The first step of their proof is to show using only differential geometry that

$$\xi \wedge \mathrm{d}\kappa = -\xi \wedge R(\xi), \tag{2.14}$$

where the 1-form $R(\xi)$ is the Ricci tensor contracted on one index with the vector ξ and the metric is used to convert the vector ξ to a 1-form. Then, one uses the Einstein equations and the dominant energy condition to show $R(\xi)$ is proportional of ξ, so the right side of 2.14 vanishes. The zeroth law then follows immediately. As with the rigidity theorems, Carter [4] independently established the zeroth law for any static or circular black hole space-time without any reference to field equations.

Bardeen, Carter and Hawking's proof of the first law takes place in two steps. The first is to derive an *integral mass formula* which expresses the mass of a black hole in terms of several of its other parameters. They consider the identity $\nabla^2 K = -2R(K)$ where ∇^2 denotes the Laplacian operator on space-time and K is any Killing vector. Integrating this identity over a partial Cauchy slice M which intersects the horizon in a 2-sphere $S_{\mathscr{H}}$ and extends to a 2-sphere S_∞ at spatial infinity yields[2]

$$\oint_{S_\infty} *\mathrm{d}K + \oint_{S_{\mathscr{H}}} *\mathrm{d}K = 8\pi G \int_M *\Big(2T(K) - \mathrm{Tr}\,[T]\,K\Big), \tag{2.15}$$

where $*$ denotes the Hodge dual operation on space-time and the Einstein equations have been used in the bulk integral. With certain choices of the Killing field, the surface integrals at infinity can be identified with various physical quantities. In particular, the surface integral at infinity for a stationary Killing field t is proportional to the total mass of space-time and that for an axial Killing field ϕ is proportional to the total angular momentum. The exact relations are given by the Komar integrals [18]

$$M := \frac{-1}{8\pi G}\oint_{S_\infty} *\mathrm{d}t \qquad \text{and} \qquad J := \frac{1}{16\pi G}\oint_{S_\infty} *\mathrm{d}\phi. \tag{2.16}$$

(Note the factor of -2 difference between the mass and angular momentum integrals.) The identity 2.15 also contains surface terms at the horizon which again can be expressed in terms of physical quantities describing the black hole. Specifically, if $\xi = t + \Omega\phi$ is the horizon-generating Killing field, we have

$$2\kappa A = \oint_{S_{\mathscr{H}}} *\mathrm{d}\xi \qquad \text{and} \qquad J_{\mathscr{H}} := \frac{-1}{16\pi G}\oint_{S_{\mathscr{H}}} *\mathrm{d}\phi, \tag{2.17}$$

where κ and A are the surface gravity and area of the horizon, respectively, and $J_{\mathscr{H}}$ can be interpreted as the angular momentum *of the horizon*. (The relative sign between the angular momenta defined by 2.16 and 2.17 is due to orientations; $S_{\mathscr{H}}$ is an *inner* boundary of M.) As one would expect, the angular momenta J and $J_{\mathscr{H}}$ generally are not equal.

[2] The orientation on $S_{\mathscr{H}}$ used here is induced by its spatial normal pointing *outward* from M, or *into* the horizon. This orientation is chosen to comply with the conventions used in the discussion of isolated horizons. The opposite orientation is often used in the literature.

Indeed, 2.15 shows their difference is given by an integral over M which is interpreted as the angular momentum contained in matter fields outside the horizon.

We are now in a position to derive *two* integral mass formulae. The first, due to Bardeen, Carter and Hawking [3], arises by applying 2.15 to the Killing field $K = t$ and the second, which is also well known [16], by choosing $K = \xi$:

$$M = \frac{\kappa A}{4\pi G} + 2\Omega J_{\mathscr{H}} - \int_M *\Big(2T(t) - \mathrm{Tr}\,[T]\,t\Big) \tag{2.18}$$

and

$$M = \frac{\kappa A}{4\pi G} + 2\Omega J - \int_M *\Big(2T(\xi) - \mathrm{Tr}\,[T]\,\xi\Big). \tag{2.19}$$

In the vacuum case, the bulk integrals in each of these formulae vanish and $J = J_{\mathscr{H}}$, whence each reduces to the mass formula found by Smarr [25] via explicit calculation with the Kerr solutions. The Einstein–Maxwell case, however, is a bit more complicated. It turns out the Maxwell stress-energy tensor $\mathbb{T}$ is trace-free and satisfies the identity

$$8\pi * \mathbb{T}(W) = (W \lrcorner\, \mathbb{F}) \wedge *\mathbb{F} - (W \lrcorner\, *\mathbb{F}) \wedge \mathbb{F}, \tag{2.20}$$

where $\mathbb{F}$ is the Maxwell field strength, W is any vector field on space-time and $\lrcorner$ denotes its contraction into the first index of a differential form. Thus, when $W = K$ is a Killing vector which Lie drags the Maxwell potential $\mathbb{A}$ as well, the bulk integral in 2.15 can be rewritten as

$$\int_M *\Big(2\mathbb{T}(K) - \mathrm{Tr}\,[\mathbb{T}]\,K\Big) = \frac{-1}{4\pi}\int_M \mathrm{d}[(K \lrcorner\, \mathbb{A}) * \mathbb{F} + \mathbb{A} \wedge (K \lrcorner\, *\mathbb{F})]. \tag{2.21}$$

Using Stokes' theorem, the right side of this expression yields a pair of surface integrals and, moreover, the term at infinity vanishes because the Maxwell potential falls off to zero there. When $K = \xi$, the horizon integral can be expressed as the product of the electric charge and electric potential of the horizon. Thus, 2.19 yields a Smarr-like formula which includes a contribution from the Maxwell field. That contribution physically represents the energy contained in the electric field outside the horizon and *should* be included in the mass M measured at infinity. For the case where $K = t$, one breaks the horizon integral from 2.21 into two pieces arising from the decomposition of t into its components along ξ and ϕ. The piece associated with ξ, as before, represents the energy in the Maxwell field outside the horizon. The piece associated with ϕ, by analogy, should represent the *angular momentum* carried by the Maxwell field outside the horizon. Indeed, using 2.21 in 2.15 with $K = \phi$, one finds

$$J = J_{\mathscr{H}} - \frac{1}{4\pi} \oint_{S_{\mathscr{H}}} (\phi \lrcorner \mathbb{A}) * \mathbb{F}, \tag{2.22}$$

where we have assumed ϕ is tangent to $S_{\mathscr{H}}$. It may seem surprising at first that the angular momentum of the bulk Maxwell field can be encoded in a surface integral at the horizon. However, the situation is directly analogous to the bulk electro-static energy being encoded in the surface term ΦQ; both arise due to the "rigidity" imposed on the problem by the presence of Killing fields. Finally, using the relation 2.22, the mass formula 2.18 reduces to the same Smarr-like expression found from 2.19. Note the angular momentum in this Smarr formula is measured *at infinity*.

The second step to Bardeen, Carter and Hawking's proof of the first law of black hole mechanics consists of varying 2.18 within the class of stationary black hole solutions and using differential geometry to evaluate the terms involving $\delta\kappa$ and $\delta\Omega$. The later calculations are rather long [3, 16] and so will not be reproduced here, but one ultimately finds the *differential mass formula*

$$\delta M = \frac{\kappa}{8\pi G}\,\delta A + \Omega\,\delta J_{\mathscr{H}} + \frac{1}{2}\int_M (T^{ab}\delta g_{ab}) * t - \delta \int_M *T(t). \tag{2.23}$$

For a wide variety of matter fields, the bulk integrals here can be calculated in terms of certain "extrinsic parameters" of the black hole to yield the first law. In the Einstein–Maxwell case, in particular, the bulk integrals will yield two terms. The first will be the electro-static "work" term $\Phi\,\delta Q$ appearing in the first law 2.11 derived by Bekenstein. The second will encode the work done by changing the angular momentum of the bulk Maxwell field. It can be calculated as $\Omega\delta(J - J_{\mathscr{H}})$ and, with this result, the differential mass formula 2.23 reduces to the standard first law 2.11 in the Einstein–Maxwell case. Note the mass and angular momentum in this expression are those measured *at infinity*.

2.4 SUMMARY AND CRITIQUE

As we have seen in this chapter, there exist a set of laws of black hole mechanics which are closely analogous to the ordinary laws of thermodynamics. While the former arise as geometric identities in general relativity, the later are due to the statistical properties of complex systems. This fact makes the analogy all the more intriguing. However, even the simplest analyses of the statistical entropy of a black hole imply it must originate in quantum mechanics. On the other hand, it seems unlikely the geometric arguments used to construct the laws of black hole mechanics could easily be carried over to a theory of quantum gravity; they depend too much on the structure of certain smooth, stationary space-times. It is therefore of considerable interest to find a formulation of black hole mechanics which is more amenable to quantization.

Even from the viewpoint of classical physics, the formulation of black hole mechanics in terms of stationary space-times is severely limited. Several of these limitations have been discussed already in the Introduction. They center around the problem that physically realistic black-hole space-times are not expected to be stationary. Consider, as an example, the situation depicted in figure 2.1 wherein a star undergoes gravitational collapse to form a non-rotating black hole. One expects the early stages of the collapse will produce a large amount of gravitational and other radiation. Eventually, however, an event horizon will form and it will grow to enclose the entire star. At first, some of the radiation from the collapse may scatter back through the event horizon, but numerical calculationssuggest these back-scattering effects will be rather short-lived and the horizon will quickly settle down to an equilibrium state. That is, during the late stages of its evolution, corresponding to the region Δ of the horizon in the figure, the black hole will be isolated from infalling radiation. Nevertheless, there are fundamental obstructions to the application of the traditional framework for black hole mechanics to this situation. Specifically, the mass arising in 2.15 is explicitly evaluated at infinity; it represents the total mass of space-time. It therefore includes the black hole mass *and* the energy contained in radiation which propagates out to null infinity and has nothing to do with the final state of the black hole. A second problem which was discussed in the Introduction has to do with the definition of surface gravity. Since it is a null surface, Δ is still generated by some null vector field ℓ. However, since ℓ is null, it can be freely rescaled by an arbitrary (positive) function on Δ, whence its acceleration — the putative surface gravity — is not uniquely defined. Even *if* ℓ can be extended to a static Killing field in a space-time neighborhood of Δ, that Killing field is defined only up to an overall, constant rescaling. Once again, the surface gravity is not unique. These problems indicate the standard framework must be extended to allow for physically reasonable situations such as in figure 2.1.

One such extension has previously been formulated by Iyer and Wald [26, 27]. Their calculations consider an arbitrary theory of gravity defined by a Lagrangian of the form

$$L = L(g_{ab}; R_{abcd}, \nabla_m R_{abcd}, \ldots; \psi, \nabla_m \psi, \ldots), \tag{2.24}$$

where R_{abcd} is the Riemann curvature of the connection ∇_m associated with g_{ab} and ψ denotes any matter fields in the theory. Consider a stationary black hole solution to the field

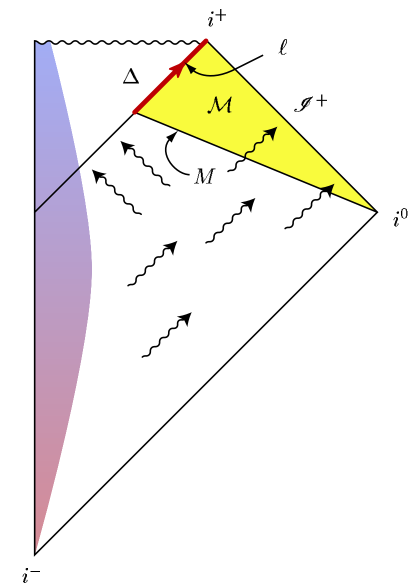

Figure 2.1: A star collapses to form a (non-rotating) black hole. The portion Δ of the event horizon at late times is isolated. The space-time $\mathcal{M}$ of interest is the triangular region bounded by Δ, $\mathscr{I}^+$ and a partial Cauchy slice M.

equations of this theory and an *arbitrary* (i.e., not necessarily stationary) variation thereof. Using general (multi-)symplectic techniques in field theory, they construct a formula for the entropy of the black hole and show it satisfies the first law of black hole mechanics under arbitrary variations. Note that to make this extension of the first law meaningful, one must carry the definitions of entropy and the other "extrinsic parameters" over to the varied solutions which themselves are only approximately stationary. Wald and Iyer accomplish this by exploiting certain properties of the bifurcation surface[3] in the unperturbed, stationary solutions. Their extension of the first law rests on the construction of the Noether charge corresponding to a diffeomorphism along the horizon-generating Killing field. The Noether charge is, by definition, the surface term in the Hamiltonian generating infinitesimal diffeomorphisms along that Killing field. That Hamiltonian has surface terms both at the horizon and at infinity, but its bulk terms vanish as usually is the case in generally covariant theories. Iyer and Wald interpret the surface term at infinity as the mass of the black hole and the term at the horizon is related to the entropy of the black hole. The first law then follows from a general identity involving the Noether charge and the (multi-)symplectic potential.

Iyer and Wald's results are very nice in that they extend the laws of black hole mechanics to an arbitrary theory of gravity and allow for non-stationary variations. Moreover, their close relation to the Hamiltonian formalism for field theory suggests the possibility of adapting them to the quantum case through canonical quantization. On the other hand, the background space-times in this formalism must still be stationary and, as before, black hole mass is calculated at infinity. One can argue the asymptotic final state of a black hole should be approximately stationary and therefore that these restrictions are not physically relevant. But there exist observables — such as the energy radiated through null infinity — which are manifestly finite in cases like that of figure 2.1, but which vanish in all stationary solutions. The existence of such observables make it unclear in what sense that black hole is approximately stationary.

The discussion of this section suggests a number of features one would like to incorporate

[3] A *bifurcate* Killing horizon consists of a pair of Killing horizons which (a) are generated by the same Killing field on space-time and (b) intersect in a two-dimensional space-like surface $\mathscr{B}$. The surface $\mathscr{B}$ is known as the *bifurcation* surface. It has been shown [28] that each stationary black hole space-time whose event horizon (a) is a Killing horizon, (b) has compact cross-sections and (c) has non-zero surface gravity can be globally extended to a new stationary space-time admitting a bifurcate Killing horizon. Moreover, the image of the event horizon in the original space-time is a proper subset of the bifurcate Killing horizon in the extended space-time. Since the equations of motion are not used in this proof, it supports the view that stationary black holes generically possess bifurcate horizons in *arbitrary* theories of gravity. Without recourse to the equations of motion, however, note the black hole's event horizon is only known to be a Killing horizon when space-time is either static or circular [4].

in a formulation of the laws of black hole mechanics. First, it seems necessary to replace the requirement of stationarity with something less restrictive in order to accommodate a broader class of physically realistic space-times such as those of figures 1.1 and 2.1. One seeks space-times describing *isolated* black holes which are not necessarily globally stationary. Second, to formulate the laws in this broader context, a definition of black hole mass should be made which does not include contributions due to radiative fields far from the horizon. Intuitively, one expects such a definition would have to be (quasi-)local to the horizon. Third, it would be highly desirable for the formulation to make contact with quantum mechanics. From a purely classical point of view, a first step in this direction would be to have a canonical (i.e., symplectic or phase space, rather than geometric or space-time) picture of the classical theory. These are the motivating principles underlying the notion of an isolated horizon. We shall see in the following chapters that all of these properties are realized in that framework.

Chapter 3

Isolated Horizons

In this chapter we specify the boundary conditions which define an isolated horizon and examine a number of examples. As explained in the Introduction, the purpose of these conditions is to model the essential features of the event horizon of a stationary black hole using only the intrinsic structure available at the horizon. In particular, we make no reference either to infinity or to a stationary Killing field.

3.1 DEFINITION

An *isolated horizon* consists of a pair $(\Delta, [\ell])$, where Δ is a three-dimensional submanifold[1] of space-time and $[\ell]$ is an equivalence class of vector fields on Δ defined up to an overall, positive rescaling:

$$\ell \sim \ell' = c\ell \quad \text{with } c \text{ a positive constant.} \tag{3.1}$$

The physical fields must satisfy the following boundary conditions at Δ:

(I) *Δ is null and $[\ell^a]$ lies along its future-directed null normal.*

(II) *Δ is expansion-free.*

This condition implies any representative $\ell \in [\ell]$ is a symmetry of the intrinsic horizon geometry in the sense

$$\mathscr{L}_\ell\, q = 0, \tag{3.2}$$

[1] In this thesis, we only consider isolated horizons with topology $S^2 \times \mathbb{R}$ since one expects black holes arising from gravitational collapse to be of this type. However, this restriction can be lifted. In particular, the analysis presented here should extend, virtually unchanged, to isolated horizons with compact cross-sections of higher genus. The extension to horizons with non-compact cross-sections (such as certain acceleration horizons) or with more complicated topology (such as those with NUT charge which have the S^3 topology) may be somewhat more subtle. These extensions will be discussed elsewhere.

where $q = \underleftarrow{g}$ is the pull-back of the space-time metric to Δ. This consequence of the boundary conditions will be discussed in more detail in the next chapter, but bears mentioning here since it motivates the next condition.

(III) *Any representative $\ell^a \in [\ell^a]$ is a (partial) symmetry of the space-time connection at Δ in the sense*

$$\mathscr{L}_\ell (\nabla_X Y) - \nabla_{\mathscr{L}_\ell X} Y - \nabla_X (\mathscr{L}_\ell Y) = 0, \tag{3.3}$$

where X and Y are arbitrary vector fields tangent to Δ.

(IV) *The equations of motion hold at Δ.*

This includes *all* components, not just the pull-backs to Δ, of all the coupled gravity-matter field equations.

(V) *Any matter fields present at the horizon satisfy conditions such that the following two properties are guaranteed:*

Va. *The vector $k := -T(\ell)$ is causal, i.e., future-directed and either time-like or null.*

This is a mild energy condition which restricts the *types* of matter present at the horizon.

Vb. *The total gravity-matter action must be differentiable.*

This is a restriction on the *boundary conditions* which may be applied to those matter fields at Δ.

The only matter explicitly considered in this thesis is the Maxwell field. We will see below that the stress-energy tensor of the Maxwell field satisfies condition Va. We will also see the property Vb is guaranteed by the following boundary condition:

($\mathrm{V}_{\mathrm{Max}}$) *Any $\ell \in [\ell]$ Lie drags the intrinsic Maxwell connection $\underleftarrow{\mathbb{A}}$.*

The constituent elements $(\Delta, [\ell])$ of an isolated horizon mimic the intrinsic structure available on a Killing horizon. Recall a Killing horizon has the property that its null normal can be scaled to coincide with a Killing vector in space-time. However, the normalization of that Killing field cannot be fixed through its properties at the horizon alone and, in the context of stationary black holes, one must refer to infinity to fix it uniquely. However, even if the scaling is not unique, the ambiguity is relatively tame: a Killing vector can be rescaled at most by a constant. Thus, a Killing horizon generically has the property that its null normal can be fixed up to an overall constant rescaling by demanding it agree with a Killing field. From a purely geometric point of view, this shows the *intrinsic* structure of a Killing horizon consists of a three-dimensional manifold and a vector field on that manifold fixed only up to constant rescalings. We have carried this essential structure over to the definition of an isolated horizon. The purpose of boundary condition I, then, is to tie the isolated horizon structure to the causal structure of space-time in much the same way as in the case of a Killing horizon. Note, however, that *a generic isolated horizon is not a Killing horizon*; $\ell \in [\ell]$ generally cannot be extended to a Killing vector in *any* space-time neighborhood of Δ. Nevertheless, it will be useful to keep this intuitive analogy in mind during the discussion of the remaining boundary conditions to follow.

Since our goal is to model an isolated, non-dynamical black hole, the reasoning behind condition II is straightforward. It simply asks that the horizon geometry — and, in particular, its area — be "time-independent." In this sense, condition II incorporates the idea that the horizon is isolated without assuming the existence of a Killing field. We will denote the area of the horizon by A_Δ and its geometric radius by R_Δ (i.e., $A_\Delta =: 4\pi R_\Delta^2$). Both are independent of the cross-section of the horizon used in their evaluation.

Condition III is somewhat more subtle than the first two. First, for motivation, let us consider the case where $(\Delta, [\ell])$ *is* actually a Killing horizon. When $\ell \in [\ell]$ can be extended to a Killing field in a space-time neighborhood of Δ, that extension must also be a symmetry of the space-time connection derived from the metric. Mathematically, this means 3.3 would apply for *all* space-time vector fields X and Y in a neighborhood of Δ. For a generic isolated horizon, ℓ cannot be so extended. In this context, condition III requires that $[\ell]$ still be a symmetry of the connection at Δ, though in a more limited sense. In particular, 3.3 is only imposed *at* Δ, and then only for vector fields X and Y which are tangent to Δ. In a sense, the situation here is similar to that normally encountered at infinity where asymptotic expansions are used to enforce symmetries on the physical degrees of freedom "up to a certain order." As mentioned previously condition II implies the horizon metric is symmetric along ℓ, a "zeroth-order" symmetry. Condition III extends that symmetry to first-order derivatives of the metric at the horizon.

A second property of condition III has to do with the geometric significance of the equivalence class $[\ell]$. A generic null hypersurface Δ does not come equipped with an equivalence class of vectors under constant rescalings such as $[\ell]$. Instead, the future-directed null normal is defined up to rescalings by arbitrary positive *functions* on Δ:

$$\ell \sim \ell' = f\ell \quad \text{with } f > 0. \tag{3.4}$$

We shall denote this equivalence class of *all* null normal fields by $\{\ell\}$. On a Killing horizon, the more restricted class $[\ell] \subset \{\ell\}$ is singled out by the group of isometries of space-time. However, this strategy will not work for a generic isolated horizon space-time which may admit *no* Killing vectors. This raises the question of how one singles out the equivalence class $[\ell]$ geometrically on an isolated horizon. The answer lies in condition III. All of the other boundary conditions (with the exception of the Maxwell field condition $\mathrm{V}_{\mathrm{Max}}$, see below) are independent of the choice of $\ell \in \{\ell\}$, but condition III is not. One can use 3.3 to analyze the question of whether the isolated horizon structure $(\Delta, [\ell])$ of a given horizon is unique. In other words, given an isolated horizon $(\Delta, [\ell])$, does there exist a non-constant, positive function f on Δ such that $(\Delta, [f\ell])$ is again an isolated horizon? The answer is almost always in the negative [10] — the isolated horizon structure is generically *unique*. In particular, the event horizon of a Kerr–Newman black hole is an isolated horizon if and only if $[\ell]$ is chosen to agree with the horizon-generating Killing field as one might expect. Note that a generic null surface Δ, even if it satisfies all our boundary conditions but condition III, may not admit a single isolated horizon structure. Work is in progress to identify some additional conditions which would guarantee the existence of an isolated horizon structure on such a surface. Luckily, the discussion of black hole mechanics is completely independent of this issue.

Condition IV is a typical dynamical boundary condition, completely analogous to that usually imposed at null infinity. Any set of boundary conditions must at least be *consistent* with the equations of motion at the horizon. Here, we simplify matters by considering only those histories where the equations of motion are satisfied at Δ from the outset. It may be possible to weaken this condition somewhat, e.g., by requiring only the pull-backs of the equations of motion at Δ. However, the exact form of a weaker version of this condition would be fairly delicate since different, but equivalent, formulations of the bulk equations of motion can have inequivalent sets of consequences when pulled-back to Δ. To avoid this subtlety, we have chosen simply to demand the full equations of motion, though only at the the points of Δ.

Finally, condition V gives the general principles which apply to incorporating matter fields at an isolated horizon and condition $\mathrm{V}_{\mathrm{Max}}$ specializes these principles to the case of a Maxwell field. The first general principle, condition Va, is a very weak energy condition. In particular, it follows immediately from the (much stronger) dominant energy condition which demands that $-T(k)$ be causal for *any* casual vector k. As we discussed in the previous chapter, the dominant energy condition is assumed by the usual derivations of the laws of black hole mechanics in the stationary context. Furthermore, since the Maxwell field satisfies the dominant energy condition, it satisfies condition Va as well. The second general principle, condition Vb, is closely tied to our use of the Hamiltonian formulation in deriving the horizon mass in the following chapters. One is generally free to satisfy this condition any way one sees fit, though some choices may be better than others. In the Maxwell case, condition $\mathrm{V_{Max}}$ is highly desirable from a geometric perspective since requires the electromagnetic connection to admit the same symmetry along $[\ell]$ as the space-time connection. We will also see below that condition $\mathrm{V_{Max}}$ is sufficient to guarantee the differentiability of the action required by condition Vb.

We conclude this section with a remark concerning the application of these boundary conditions to specific classes of black holes. The form of the boundary conditions given here represent the most general definition of an isolated horizon. In practice, one may want to model isolated horizons which correspond to specific classes of stationary black holes with certain additional symmetries. In such cases the conditions given here may be supplemented by additional conditions which implement the specific symmetries desired. Later in this thesis, since we will be concerned with isolated horizons analogous to the Kerr–Newman black holes, we will consider isolated horizons with a specific axial symmetry. In previous papers [6, 7], we considered the analogs of Reissner–Nordström holes. There, we imposed conditions guaranteeing the spherical symmetry of the metric and of the electro-magnetic flux densities at the horizon as befit the situation we were modelling.

3.2 EXAMPLES

It is easy to check that any Killing horizon is an isolated horizon when $[\ell]$ is defined using the horizon-generating Killing field. In particular, the event horizon of a Kerr–Newman black hole is an isolated horizon when $[\ell]$ is defined in the usual way. One can define similar stationary black holes in space-times with non-zero cosmological constant and, again, their event horizons are naturally isolated horizons. Finally, it has been shown [7] that cosmological horizons in de Sitter space-time are also isolated horizons. These examples represent the bulk of the cases usually considered in the stationary framework for black hole mechanics. Thus, the isolated horizon framework will incorporate the known results.

Isolated horizons are also much more general than the traditional, globally stationary black hole event horizons. Consider the spherical collapse of figure 2.1. Physically, one expects the geometry near the horizon Δ at asymptotically late times to be (approximately) isometric to a stationary solution, though the geometry far away will describe gravitational radiation propagating toward infinity. It follows that Δ will be an isolated horizon. Likewise, one expects the surfaces Δ_1 and Δ_2 in figure 1.1 will be isolated horizons on similar physical grounds. More rigorously, an infinite-dimensional space of examples can be constructed by starting with Killing horizons and "adding radiation." To be specific, consider one asymptotic region of a stationary black hole space-time and a partial Cauchy surface M therein. The idea is to define a new set of initial data on this slice which, however, agrees with the original set in some compact neighborhood of the horizon. In the Einstein–Maxwell case, one can accomplish this using the strategy introduced by Cutler and Wald [29] in their proof of existence of solutions with smooth null infinity. By this technique, one constructs a space-time region $\mathcal{M}$ bounded by M, Δ and a second partial Cauchy slice M' to the future of M in which Δ is an isolated horizon (see figure 3.1). Due to the presence of radiation, $\mathcal{M}$ will not admit any global Killing field, though it will admit at least the horizon-generating Killing field in a neighborhood of Δ.

We now construct a family of space-times containing isolated horizons which are *not* generated by (local) Killing fields. Currently, these examples exist only in the non-rotating context, though work is under way to extend this construction to the rotating case [10]. In this limited context, however, an infinite-dimensional space of such horizons can be constructed using Friedrich's results [30], and Rendall's extension [31] thereof, on the null initial value formulation (see figure 3.2) of general relativity. In this framework, one considers two null hypersurfaces, Δ and $\mathscr{N}$, which intersect in a 2-sphere S (see figure 3.2). One chooses a null tetrad $(\ell, n, m, \bar{m})$ in a neighborhood of these surfaces such that ℓ and n are normal to Δ and $\mathscr{N}$, respectively. Using this tetrad, one can contemplate the Newman–Penrose formulation (see Appendix B) of the gravitational field equations in a space-time neighborhood of Δ and $\mathscr{N}$. In a suitable choice of gauge [30], the free data for the vacuum Einstein equations consists of Ψ_0 on Δ, Ψ_4 on $\mathscr{N}$, and the intrinsic metric ${}^2g$ as well as the Newman–Penrose coefficients λ, σ, π, $\mathrm{Re}\,[\mu]$ and $\mathrm{Re}\,[\rho]$ on the two-sphere S. Given these fields, there is a unique solution (modulo diffeomorphisms) to the vacuum Einstein equations in a neighborhood of S bounded by (and including) the appropriate portions of Δ and $\mathscr{N}$. If we set $\Psi_0 = 0$ on Δ and $\rho = \sigma = 0$ on S, we guarantee all the isolated horizon boundary conditions but condition III will hold on Δ. In the non-rotating context [7, 9], we require further that $\pi = 0$ on S (see 4.28), thereby guaranteeing boundary condition III as well. Lewandowski has shown [32] that, in the resulting solution, Δ is a non-rotating isolated horizon. Note that Ψ_4 need not vanish *anywhere* in the space-time region rele-

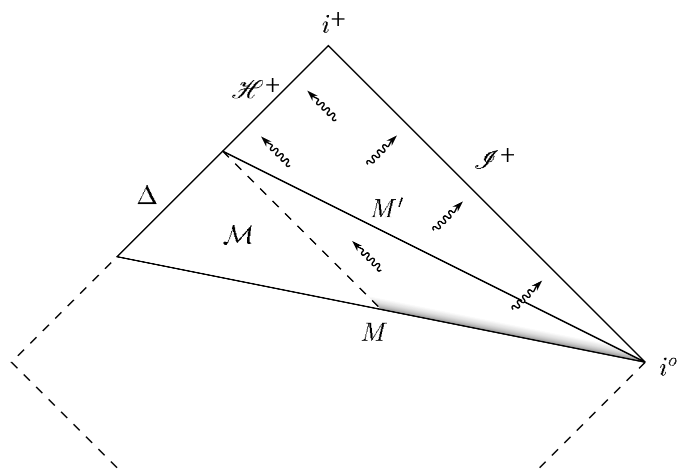

Figure 3.1: A space-time $\mathcal{M}$ with an isolated horizon Δ as internal boundary and radiation field in the exterior can be obtained by starting with an asymptotic region of a stationary black hole space-time and modifying the initial data on the partial Cauchy surface M. While the new metric continues to be isometric with the original metric in a neighborhood of Δ, it admits radiation in a neighborhood of infinity. The dashed lines refer to the original asymptotic region.

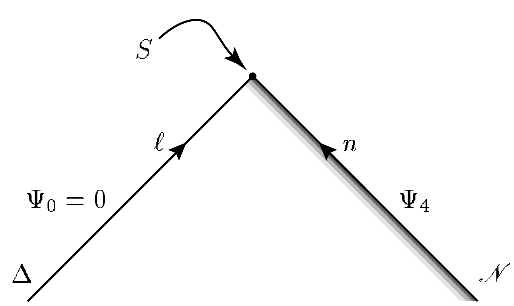

Figure 3.2: Space-times with isolated horizons can be constructed by solving the characteristic initial value problem on two intersecting null surfaces, Δ and $\mathscr{N}$. The final solution admits Δ as an isolated horizon. Generically, there is radiation arbitrarily close to Δ and no Killing fields in any neighborhood of Δ.

vant to this construction. Thus, in the vacuum case, there is an infinite-dimensional space of (local) solutions admitting non-rotating isolated horizons at Δ. One can show, in this setting, that there always exists a vector field ξ in a neighborhood of Δ which lies along the null normal to the horizon and satisfies $\mathscr{L}_\xi\, g \,\hat{=}\, 0$ (note the metric is *not* pulled back to Δ). However, the Weyl curvature of the metric g is generally *not* Lie dragged along ξ, even *at* the horizon itself. Thus, ξ cannot be a Killing field for the space-time metric in *any* neighborhood of Δ. (See [32] for details.) The Robinson–Trautman space-times provide an interesting class of exact solutions which bring out this point [33]: a sub-class of these solutions admit an isolated horizon, but no Killing fields whatsoever. There is radiation in every neighborhood of the isolated horizon in these solutions. However, in a natural chart, the metric coefficients and several of their radial derivatives evaluated *at* Δ are the same as those of the Schwarzschild metric at its event horizon.

The constructions depicted in figures 3.1 and 3.2 are complementary. The first defines a space-time which extends from the isolated horizon to infinity, but in which there is no radiation in a neighborhood of Δ. The second, on the other hand, constructs only a space-time neighborhood of the horizon, but allows arbitrary radiation fields near or even *at* the horizon, provided there is no flux *across* the horizon, of course. We expect there will exist an infinite-dimensional space of solutions to the vacuum Einstein equations (as well as to the Einstein–Maxwell equations) which are free from both limitations. In other words, these solutions will extend to spatial infinity and admit isolated horizons with radiation arbitrarily close to them. However, a comprehensive treatment of this issue will be technically difficult. Given the current status of global existence and uniqueness results in the asymptotically flat context, the present limitations are not surprising. Indeed, the situation at null infinity is somewhat analogous: while the known techniques have provided several interesting partial results, they do not yet allow us to show there exists a large class of solutions to Einstein's vacuum equations which admit complete and smooth past and future null infinities, $\mathscr{I}^\pm$ and the standard structure at spatial infinity, i^o.

Chapter 4

The General Structure of Isolated Horizons

In this chapter, we explore the consequences of the boundary conditions stated above. We begin by recalling some facts concerning the geometric structure of null surfaces. Then, using the language introduced in that discussion, we will examine the basic geometry of an isolated horizon. We will then introduce matter, specifically the Maxwell field, at the horizon, and analyze the consequences of the boundary conditions for the matter field. Finally, we will discuss a unique, canonical foliation which exists for every (non-extremal) isolated horizon. This discussion will be particularly useful in the action formulation of an isolated horizon system in the next chapter.

4.1 THE GEOMETRY OF NULL SURFACES

We begin our analysis of the structure of isolated horizons by recalling some facts about null hypersurfaces in space-time. These surfaces possess several features which are not found in space-like or time-like hypersurfaces. Our purpose here is to outline some of these features and to introduce some notation which will be useful in the following sections.

Consider a null submanifold Δ of a Lorentzian space-time $\mathcal{M}$. Space-time vectors normal to a null hypersurface are also tangent to it and lie in the unique degenerate direction of the induced metric $q := \underleftarrow{g}$. Since these vectors are null, the future-directed normal to Δ can only be defined as a *direction* field $\{\ell\}$. As before, $\{\ell\}$ denotes an equivalence class of vector fields on Δ under the equivalence relation 3.4.

Since $\{\ell\}$ is everywhere orthogonal to Δ, any $\ell \in \{\ell\}$ is automatically geodetic:

$$\nabla_\ell \ell = \kappa_\Delta^{(\ell)} \ell \qquad (4.1)$$

for some function $\kappa_\Delta^{(\ell)}$ on Δ which depends on the particular null normal field chosen. Thus, Δ is ruled by a null geodesic congruence. To study this congruence, we consider the projective geometry of Δ. That is, we will study the geometry of the two-dimensional manifold $\mathscr{P}\Delta$ of orbits of $\{\ell\}$ in Δ. By its very definition, $\mathscr{P}\Delta$ is the base space for an affine line bundle whose total space is Δ. The projection $\Gamma : \Delta \to \mathscr{P}\Delta$ for this bundle is given by mapping a point of the horizon to the null geodesic on which it lies. The differential $D\Gamma$ of the projection maps tangent vectors on Δ to tangent vectors on $\mathscr{P}\Delta$ and the kernel of this map is precisely the vertical direction $\{\ell\}$. Thus, a tangent vector on $\mathscr{P}\Delta$ can be realized as an equivalence class of vectors on Δ where vectors which differ by an element of $\{\ell\}$ are equivalent. The equivalence class associated to a vector V will be denoted either by $\mathscr{P}(V)$ or, more often, by $\widehat{V}$. Note that, by taking its equivalence class, *any* vector on Δ gives rise to a vector on $\mathscr{P}\Delta$. This is *not* the case with covectors. Viewed as a linear map on the tangent space, $\omega \in T^*_p\Delta$ will give rise to an element of $T^*_{\Gamma(p)}\mathscr{P}\Delta$ if and only if it vanishes on the kernel $\{\ell\}$ of $D\Gamma$. Covectors with this property will be called *projectable.* The projection of a projectable covector ω will again be denoted either by $\mathscr{P}(\omega)$ or, more often, by $\widehat{\omega}$. The situation with higher valence tensors is similar to that with covectors. It turns out that a (space-time) tensor $T_{a_1\cdots a_m}{}^{b_1\cdots b_n}$ on Δ gives rise to a tensor $\widehat{T}_{a_1\cdots a_m}{}^{b_1\cdots b_n}$ on $\mathscr{P}\Delta$ if and only if the contraction of ℓ^{a_i} or ℓ_{b_j} on any index of T yields a sum of terms, each of which is proportional either to ℓ^{a_p} or ℓ_{b_q}. Tensors with this property will again be called *projectable.*

We now have a map $\widehat{\cdot}$ which takes certain tensors at points of Δ to tensors at points of $\mathscr{P}\Delta$. To proceed, we must extend this map to tensor *fields.* A tensor field T on Δ will give rise to a tensor field $\widehat{T}$ on $\mathscr{P}\Delta$ if and only if (i) T is projectable at every point of Δ and (ii) $\widehat{\mathscr{L}_\ell T} = 0$. The second condition here simply requires that the tensors on $\mathscr{P}\Delta$ gotten from projecting the tensors at different points along a given generator in Δ all agree. Note this second condition is also independent of the choice of $\ell \in \{\ell\}$. Unfortunately, condition (ii) is sometimes violated by certain tensors of physical interest. To remedy this situation, we must allow "time-dependent" tensor fields on $\mathscr{P}\Delta$ arising from general tensor fields on Δ which are projectable at each point, but do not satisfy condition (ii). To make this notion precise, pick a cross-section of Δ and a representative $\ell \in \{\ell\}$. We then get a map $\Delta \to \mathscr{P}\Delta \times \mathbb{R}$ given by $p \mapsto (\Gamma(p), \lambda)$, where λ is the affine distance along ℓ from the initial cross-section of Δ to p. Tensors which are everywhere projectable, but fail to satisfy condition (ii) give rise to tensors on $\mathscr{P}\Delta$ which depend on the "time" parameter λ.

Let us now apply the above constructions to our analysis of the null congruence generated by $\{\ell\}$. The first thing we note is that the space-time metric g is projectable at each point of Δ and gives rise to a "time-dependent" Riemannian metric on $\mathscr{P}\Delta$. We denote this metric by $\widehat{q}$ since its pull-back to Δ under Γ gives the degenerate intrinsic metric q. To see that $\widehat{q}$ is indeed "time-dependent," pick an $\ell \in \{\ell\}$ and form the tensor

$$B^{(\ell)}_{ab} := \nabla_a \ell_b \quad \Rightarrow \quad \mathscr{L}_\ell\, g_{ab} = B^{(\ell)}_{(ab)}. \tag{4.2}$$

It easy to show $B^{(\ell)}_{ab}$ projects to a tensor $\widehat{B}^{(\ell)}_{ab}$ on $\mathscr{P}\Delta$. This projected tensor is conventionally decomposed into pieces representing the expansion $\theta^{(\ell)}$, shear $\widehat{\sigma}^{(\ell)}_{ab}$ and twist $\widehat{\omega}^{(\ell)}_{ab}$ of ℓ according to

$$\widehat{B}^{(\ell)}_{ab} =: \tfrac{1}{2}\theta^{(\ell)}\widehat{q}_{ab} + \widehat{\sigma}^{(\ell)}_{ab} - \widehat{\omega}^{(\ell)}_{ab}, \tag{4.3}$$

where $\widehat{\sigma}^{(\ell)}_{ab}$ is symmetric and trace-free and $\widehat{\omega}^{(\ell)}_{ab}$ is anti-symmetric. Note that, since ℓ is hypersurface-orthogonal (to Δ), the pull-back of $B^{(\ell)}_{[ab]}$ to Δ vanishes, whence the twist of ℓ is zero. However, there is no reason to expect the expansion and shear of ℓ will vanish, so $\widehat{q}$ is generically "time-dependent." Furthermore, the expansion and shear of ℓ may themselves by "time-dependent." To see this, note that $B^{(\ell)}_{ab}$ satisfies

$$\mathscr{L}_\ell B^{(\ell)}_{ab} = \kappa^{(\ell)}_\Delta B^{(\ell)}_{ab} + B^{(\ell)}_{ac} B^{(\ell)}{}_b{}^c + \ell^c R_{cab}{}^d \ell_d + \ell_b \nabla_a \kappa^{(\ell)}_\Delta, \tag{4.4}$$

where $R_{abc}{}^d$ is the space-time Riemann tensor. Again, the right side of this equation is projectable, but generally does not project to zero. Nevertheless, projecting both sides of 4.4 yields a useful pair of "evolution" equations[1] for the expansion and shear of ℓ:

$$\tfrac{d}{d\lambda}\,\theta^{(\ell)} = \kappa^{(\ell)}_\Delta \theta^{(\ell)} - \tfrac{1}{2}\left(\theta^{(\ell)}\right)^2 - \widehat{\sigma}^{(\ell)}_{ab}\widehat{\sigma}^{(\ell)ab} + \widehat{\omega}^{(\ell)}_{ab}\widehat{\omega}^{(\ell)ab} - R_{cd}\ell^c\ell^d, \tag{4.5}$$

$$\tfrac{d}{d\lambda}\,\widehat{\sigma}^{(\ell)}_{ab} = \kappa^{(\ell)}_\Delta \widehat{\sigma}^{(\ell)}_{ab} - \theta^{(\ell)}\widehat{\sigma}^{(\ell)}_{ab} + \mathscr{P}(C_{cabd}\ell^c\ell^d), \tag{4.6}$$

where R_{cd} is the Ricci tensor and C_{cabd} the Weyl tensor on space-time. The first of these results, 4.5, is the well-known Raychaudhuri equation, generalized to null congruences. We could also have derived a third equation describing the "evolution" of the twist, but this equation is satisfied identically in our case since the twist vanishes. Finally, note that when ℓ is rescaled by a function f as in 3.4, the expansion and shear are simply rescaled by the same function.

Let us now examine the issue of covariant differentiation on the null surface Δ. Recall that, in the case of a space-like or time-like surface, there is a unique covariant derivative which is compatible with the (non-degenerate) metric induced on the surface by the space-time metric. In the null case, the induced metric is degenerate and there are many compatible connections on Δ. However, recall again from the non-null case that there is a

[1] In deriving these expressions, we have used the result $\mathscr{P}(B^{(\ell)}_{ac}B^{(\ell)}{}_b{}^c) = \widehat{B}^{(\ell)}_{ac}\widehat{B}^{(\ell)}{}_b{}^c$. The contraction of indices does not generally commute with the projection to $\mathscr{P}\Delta$, but one can show in this case that it does.

second way to construct the intrinsic covariant derivative on the surface from its space-time analog: Take the space-time covariant derivative of one vector field which is tangent to the surface along another and project the result into the surface. The purpose of taking the projection in this procedure is to guarantee that the intrinsic covariant derivative yields a vector field which is again tangent to the submanifold. In the case of a null surface Δ, there is no natural projection of space-time vectors into Δ. However, it may happen that the space-time covariant derivative of one vector field tangent to Δ along another is always tangent to Δ anyway. This will occur precisely when

$$\nabla_{\underleftarrow{a}} \ell_b = \varpi_a^{(\ell)} \ell_b. \tag{4.7}$$

Note that, for a non-null surface, the analogous condition would be that the extrinsic curvature of the surface vanishes. In our null case, however, 4.7 is exactly equivalent to restricting ℓ^a to be both expansion- and shear-free. On such a surface, the intrinsic covariant derivatives of a vector field v^b and a covector field ω_b are given by

$$\mathscr{D}_a v^b := \nabla_{\underleftarrow{a}} V^b \qquad \text{and} \qquad \mathscr{D}_a \omega_b := \underleftarrow{\nabla_a \Omega_b}, \tag{4.8}$$

respectively. Here, V^b denotes any vector field in a space-time neighborhood of Δ whose restriction to Δ agrees with v^b, and Ω_b denotes any covector field in a space-time neighborhood of Δ such that $\Omega_{\underleftarrow{b}} = \omega_b$. One can easily check that condition 4.7 guarantees the definitions 4.8 are independent of the particular space-time extensions chosen for the fields acted upon by the connection. Moreover, 4.7 implies

$$\mathscr{D}_a \ell^b = \varpi_a^{(\ell)} \ell^b. \tag{4.9}$$

One can see from this formula that, although the connection $\mathscr{D}$ does not depend on the choice of $\ell \in \{\ell\}$, the one-form $\varpi^{(\ell)}$ does. In fact, one finds $\varpi^{(\ell)}$ transforms under a rescaling as in 3.4 as

$$\varpi^{(f\ell)} = \varpi^{(\ell)} + \mathrm{d}(\ln f). \tag{4.10}$$

However, in the case of an isolated horizon, when we restrict ℓ to $[\ell] \subset \{\ell\}$, the connection form $\varpi^{(\ell)}$ *will* be independent of the choice of ℓ and we can simply write ϖ.

4.2 ISOLATED HORIZON GEOMETRY

We will now apply the results of the previous section to the particular case of an isolated horizon $(\Delta, [\ell])$. In this discussion, we will restrict the equivalence class $\{\ell\}$ used in the previous section to the class $[\ell]$ to which the boundary conditions refer.

Boundary condition II requires the expansion of $[\ell]$ to vanish — a condition which is independent of the choice of $\ell \in [\ell]$. Under this restriction, the Raychaudhuri equation 4.5 simplifies to

$$0 = \widehat{\sigma}^{(\ell)}_{ab}\widehat{\sigma}^{(\ell)ab} + R_{cd}\ell^c\ell^d. \tag{4.11}$$

Since $\mathscr{P}\Delta$ is a Riemannian manifold, the square of the shear is non-negative. Furthermore, the energy condition Va, together with the equations of motion at Δ, guarantees the second term is also non-negative. Thus, both terms must vanish separately. It follows that the congruence generated by $[\ell]$ is shear-free and

$$R_{ab}\ell^a\ell^b = 8\pi G T_{ab}\ell^a\ell^b = 0 \quad \Rightarrow \quad T(\ell) = -e\ell \tag{4.12}$$

for some non-negative function e on Δ. Physically, this result simply states there is no flux of matter energy-momentum through the surface Δ. Furthermore, using 4.2, the vanishing expansion and shear of $[\ell]$ imply the intrinsic metric q on Δ is Lie-dragged along $[\ell]$. These two results are important reflections of the horizon's isolation.

Now let us examine the space-time curvature at the horizon. The main implication of the boundary conditions for the Ricci curvature arises by applying the Einstein equations at Δ to 4.12. This yields

$$R(\ell) = [\Lambda_0 - 4\pi G(T + 2e)]\ell, \tag{4.13}$$

where Λ_0 denotes the cosmological constant, T is the trace of the matter stress-energy tensor, and e is the function introduced in 4.12. In particular, note that this relation implies the pull-back of its left hand side to Δ vanishes. With this result in hand, we can now examine the Weyl curvature at the horizon. Since Δ is shear-free, it follows immediately from 4.6 that ℓ is a principal null direction[2] of the space-time metric at the horizon:

$$C_{c\underleftarrow{a}bd}\ell^c\ell^d = 0. \tag{4.14}$$

This result is not terribly surprising; the Weyl tensor always possesses four principal null directions and, as we have just shown, the null normal to any shear-free null hypersurface will be one of them. However, the boundary conditions actually allow the proof of the stronger statement that ℓ is a *repeated* principal null direction[3] of g at Δ. To accomplish this proof, we use 4.7 to calculate

$$R_{\underleftarrow{ab}cd}\ell^d = 2\mathscr{D}_{[a}\mathscr{D}_{b]}\ell^c = (\mathrm{d}\varpi)_{ab}\ell_c. \tag{4.15}$$

[2] In the Newman–Penrose language, this means the curvature component Ψ_0 vanishes whenever ℓ^a is chosen as the first null tetrad element. The Newman–Penrose formalism is discussed in Appendix B.

[3] In the Newman–Penrose language, this means the curvature components Ψ_0 and Ψ_1 *both* vanish whenever ℓ^a is chosen as the first null tetrad element.

If we pull this expression back on its third index as well, the right side vanishes. Then, using the trace-free character of the Weyl tensor and 4.13, it follows that ℓ is indeed a repeated principal null direction of the space-time metric at the horizon:

$$C_{c\underleftarrow{a}bd}\ell^c\ell^d = 0. \tag{4.16}$$

Note that we no longer pull-back on the third index here. Thus, every space-time admitting an isolated horizon is of Petrov type II at the horizon itself. This is an important characteristic which carries over from the stationary context, where the usual black hole solutions are of type II-II everywhere, even off the horizon. In contrast, however, note that the space-time geometry away from an isolated horizon horizon need not be algebraically special at all. Furthermore, since the definition of an isolated horizon is made locally, we have no results on the null directions transverse to Δ, whence the space-time geometry at an isolated horizon may not be of type II-II.

All of the results of this section so far have not made use of condition III. Let us now examine the additional consequences which result when it is taken into account. We have already seen the geodesic congruence generating an isolated horizon is both expansion- and shear-free, whence the connection $\mathscr{D}$ described in the previous section is defined on Δ. In terms of the intrinsic connection, condition III simply reads

$$[\mathscr{L}_\ell, \mathscr{D}_a]V^b = 0 \quad \text{for all } V^b \text{ on } \Delta. \tag{4.17}$$

This result makes the content of condition III even more transparent than before: it insists the intrinsic connection, as well as the intrinsic metric, on Δ is "Lie dragged" along ℓ. If the geometry of Δ were non-degenerate, these two properties of its geometry would not be independent. However, since $\mathscr{D}$ is not uniquely determined by q in the null case, the additional restriction is necessary here. Now, setting $V = \ell$ in 4.17 and using 4.9 immediately implies

$$\mathscr{L}_\ell \varpi = 0. \tag{4.18}$$

In view of the definition 4.7 of ϖ, the contraction $\ell \lrcorner\, \varpi$ is simply the acceleration $\kappa_\Delta^{(\ell)}$ of ℓ on Δ. In analogy with the usual definition 2.13 for a Killing horizon, we take this acceleration to define the *surface gravity* of an isolated horizon. Since we have no prescription for choosing $\ell \in [\ell]$, the surface gravity is defined only up to an overall, multiplicative constant, whence the notation $\kappa_\Delta^{(\ell)}$. Note, however, that contracting ℓ into 4.18 immediately implies the surface gravity is constant along each generator of Δ no matter which ℓ is used. Just as in the Killing horizon case, we must do more work to show the surface gravity is also constant from one generator to another.

Using the curvature constraints 4.16 and 4.13 in 4.15, we derive the new result

$$\ell \lrcorner \, \mathrm{d}\varpi = 0, \tag{4.19}$$

This result implies the two-form $\mathrm{d}\varpi$ is projectable to $\mathscr{P}\Delta$. Moreover, since all two-forms on $\mathscr{P}\Delta$ are proportional, the projection can be written as the product of some scalar function with the volume form $\hat{\epsilon}$ on $\mathscr{P}\Delta$. The two-form $\mathrm{d}\varpi$ on Δ is therefore proportional to the pull-back of $\hat{\epsilon}$ under the projection map Γ, denoted ${}^2\epsilon$. Following the Newman–Penrose notation[4], we write

$$\mathrm{d}\varpi = 2\mathrm{Im}\,[\Psi_2]\;{}^2\epsilon. \tag{4.20}$$

The result 4.18 then implies $\mathrm{Im}\,[\Psi_2]$ is constant along each generator of Δ. Furthermore, the zeroth law is now easy to prove since using 4.19 in the Cartan formula for the Lie derivative in 4.18 implies

$$\mathrm{d}(\ell \lrcorner \, \varpi) = d\kappa_\Delta^{(\ell)} = 0 \tag{4.21}$$

on Δ. This is the zeroth law of black hole mechanics for isolated horizons. Although we have not singled out the *value* of the surface gravity (since $\ell \in [\ell]$ is undetermined), we have shown all choices of $\ell \in [\ell]$ lead to a uniform surface gravity on Δ. The situation is similar with Killing horizons: no matter which horizon-generating Killing field is used to fix the scaling of the null normal to the horizon, the surface gravity is uniform.

4.3 THE FORM OF THE MAXWELL FIELD

In the previous section, we have analyzed the restrictions placed on the gravitational degrees of freedom (i.e., the space-time geometry) by the isolated horizon boundary conditions. In this section, we extend that analysis to the Maxwell field, which is the only type of matter allowed at the horizon in this thesis. For generic matter fields, the most important consequence of the boundary conditions is 4.12. However, it appears the full import of this result must be analyzed on a case-by case basis, as is done below for the Maxwell field.

Let us begin by recalling the Maxwell stress-energy tensor can be expressed in two forms:

$$\mathbb{T}_{ab} = \frac{1}{4\pi}\left[\mathbb{F}_{ac}\mathbb{F}_b{}^c - \tfrac{1}{4}g_{ab}\mathbb{F}_{cd}\mathbb{F}^{cd}\right] = \frac{1}{4\pi}\left[({*}\mathbb{F})_{ac}({*}\mathbb{F})_b{}^c - \tfrac{1}{4}g_{ab}({*}\mathbb{F})_{cd}({*}\mathbb{F})^{cd}\right], \tag{4.22}$$

[4]The Newman–Penrose components usually depend on the particular null tetrad used in their calculation. However, since the curvature components Ψ_0 and Ψ_1 vanish in our case, the value of Ψ_2 is actually independent of the choice of n^a, m^a and $\bar{m}^a$. Thus, the notation used here is consistent. See Appendix B for details.

where $\mathbb{F}$ denotes the Maxwell field strength and $*\mathbb{F}$ denotes its space-time dual. Using the first form and contracting ℓ on both indices of the stress energy tensor, 4.12 implies the vector $\ell \lrcorner \mathbb{F}$ is null. Moreover, since $\mathbb{F}$ is antisymmetric, that vector is also normal to ℓ. It follows that $\ell \lrcorner \mathbb{F}$ is proportional to ℓ, and we have

$$\ell \lrcorner \underleftarrow{\mathbb{F}} = 0 = \ell \lrcorner \underleftarrow{*\mathbb{F}}, \tag{4.23}$$

where we have used the second form of the stress energy tensor to derive the second result. These results show the Maxwell field strength and its space-time dual are both projectable to $\mathscr{P}\Delta$ at each point of Δ. We define the electric and magnetic flux densities *out* of the horizon[5] by $\mathbb{E}_\Delta := -\widehat{*\mathbb{F}}$ and $\mathbb{B}_\Delta := -\widehat{\mathbb{F}}$, respectively. The Maxwell equations imply the exterior derivatives of $\mathbb{F}$ and $*\mathbb{F}$ both vanish at the horizon, so

$$\mathscr{L}_\ell \underleftarrow{\mathbb{F}} = 0 = \mathscr{L}_\ell \underleftarrow{*\mathbb{F}}. \tag{4.24}$$

Thus, the flux density 2-forms $\mathbb{E}_\Delta$ and $\mathbb{B}_\Delta$ are both "time-independent," but are otherwise unrestricted. In particular, they need not be spherically symmetric as in [7]. Finally, the electric and magnetic charges *inside* (see footnote) the horizon are defined by

$$-4\pi \mathbb{Q}_\Delta = \oint_{S_\Delta} *\mathbb{F} \qquad \text{and} \qquad -4\pi \mathbb{P}_\Delta = \oint_{S_\Delta} \mathbb{F}, \tag{4.25}$$

where S_Δ is any spherical section of Δ. Since the flux densities are "time-independent", it follows immediately that both charges are independent of the choice of section S_Δ.

Now let us examine the consequences of condition $\mathrm{V}_{\mathrm{Max}}$ for the Maxwell potential. As we stated previously, this condition is designed to render the total Einstein–Maxwell action differentiable. However, as we will see now, it plays an additional key role in our formulation of the first law of black hole mechanics. Define the electric potential Φ_Δ of the horizon by

$$\Phi_\Delta := -\ell \lrcorner \mathbb{A} \tag{4.26}$$

on Δ. In the formulation of black hole mechanics in terms of stationary space-times, this quantity appears in the first law as the coefficient of the electric charge variation. The first law makes sense in this context only because the (global) stationarity of the Maxwell potential guarantees the electric potential is constant over the horizon. In the case of

[5] Consider a space-like hypersurface M which intersects Δ in a two-sphere S_Δ. The orientation of S_Δ is defined with respect to the spatial normal pointing *outward* from M, or *into* the horizon. We want to characterize the electro-magnetic fields arising from charges *inside* the horizon. The signs in our definitions are chosen to accommodate the orientation of S_Δ.

isolated horizons, condition V_{Max} only requires ℓ to be a symmetry of $\underleftarrow{\mathbb{A}}$ *at the horizon.* However, using 4.23, this is still sufficient to find

$$\mathscr{L}_\ell \underleftarrow{\mathbb{A}} = -\mathrm{d}\Phi_\Delta = 0. \tag{4.27}$$

Thus, the electric potential is constant over *any* isolated horizon. In the stationary context, we can use the global Killing field to find a value for the electric potential at the horizon using fall-off conditions at infinity. For general isolated horizons, there is no such prescription: the electric potential is constant, but we do not know its value. This is directly analogous to the situation with the surface gravity discussed in the previous section.

4.4 HORIZON FOLIATION

The previous sections have examined the restrictions placed by the boundary conditions on the physical fields at an isolated horizon. The purpose of this section is to explore the general structures which exist on the horizon as a result of these restrictions. Specifically, we will be concerned with the existence of a preferred foliation of a generic (non-extremal) isolated horizon by 2-spheres. The existence of this foliation has important implications for the structure of the symmetry group of the horizon and will impact the phase space construction of the next chapter.

Let us begin by describing the foliation of Δ in the non-rotating case, since its definition there is simpler. First, a *non-rotating isolated horizon* $(\Delta, [\ell])$ is defined by the boundary conditions set out in chapter 3, together with the additional restriction

$$\mathrm{Im}\,[\Psi_2] \mathrel{\widehat{=}} 0. \tag{4.28}$$

According to 4.20, this is precisely the case where ϖ is curl-free. However, ϖ has non-vanishing contraction with ℓ when the isolated horizon is non-extremal (i.e., when its surface gravity is not zero) and therefore does not vanish identically. Thus, ϖ is hypersurface-orthogonal everywhere on Δ. The surfaces to which it is orthogonal are the leaves S_Δ of the preferred foliation. Each S_Δ must be transverse to ℓ since $\ell \lrcorner\, \varpi = \kappa_\Delta^{(\ell)} \neq 0$, and therefore must have the topology of a 2-sphere.

While $\mathrm{Im}\,[\Psi_2]$ does not vanish for a general (rotating) isolated horizon, this function is still constant along each generator of Δ. Thus, the curl of ϖ in 4.20 projects to a "time-independent" 2-form $\widehat{\mathrm{d}\varpi}$ on $\mathscr{P}\Delta$. We will seek a 1-form potential[6] $\hat{\varpi}$ on $\mathscr{P}\Delta$ for the

[6] We commit a minor abuse of notation here. Note that the 1-form $\hat{\varpi}$ being introduced here is *not* the same as the projection $\widehat{\varpi}$ of the horizon connection. In fact, the later does not even exist when the surface gravity is non-zero.

projected curvature $\widehat{\mathrm{d}\varpi}$. Of course, this potential is not unique; one is free to add to it the gradient of any function on $\mathscr{P}\Delta$. However, there is a fairly natural gauge condition we will use to fix this ambiguity: we insist $\hat{\varpi}$ is divergence-free on $\mathscr{P}\Delta$. Since there are no harmonic 1-forms on the 2-sphere $\mathscr{P}\Delta$, there exists a unique 1-form $\hat{\varpi}$ such that

$$d\hat{\varpi} = 2\mathrm{Im}\,[\Psi_2]\,\hat{\epsilon} \qquad \text{and} \qquad d\hat{*}\hat{\varpi} = 0, \tag{4.29}$$

where $\hat{*}$ denotes the Hodge dual on $\mathscr{P}\Delta$. Since the right sides of both equations in 4.29 are "time-independent," the potential $\hat{\varpi}$ will be so as well. Consequently, we can pull it back under the projection map to find a 1-form $\mathring{\varpi}$ on Δ satisfying $\mathscr{L}_\ell \mathring{\varpi} = 0 = \ell \lrcorner \mathring{\varpi}$. Moreover, the curvature of $\mathring{\varpi}$ is the same as that of ϖ and we can write

$$\varpi = \mathring{\varpi} + \mathrm{d}\psi \tag{4.30}$$

for some (globally-defined) function ψ on Δ. Now, in the non-rotating case, the divergence *and* curl of $\hat{\varpi}$ vanish on $\mathscr{P}\Delta$, whence $\mathring{\varpi} = 0$ on Δ in that case. Thus, the function ψ is constant on each leaf of the preferred foliation defined above for a non-rotating horizon. We will carry this definition over to the general, rotating case: the leaves S_Δ of the preferred foliation of Δ are the level surfaces of the function ψ.

The attitude adopted in this thesis is that the isolated horizon boundary conditions represent *restrictions* on the usual physical fields; there are no new degrees of freedom at the horizon. In this spirit, the definition of the function ψ of 4.30 requires a little more care. The problem is the definition, as it stands, allows for the addition of an arbitrary constant to ψ without changing any of the present results. To fix this ambiguity, note that the geodesic generators of an isolated horizon are typically past-incomplete and terminate on some 2-sphere cross-section S_Δ^- of the horizon. From this point on, we require not only that such a surface exists, but also that it is one of the leaves of the preferred foliation of Δ. In the next chapters on the action and phase space constructions, we will *always* work with a *finite* (in affine length along ℓ) segment of an isolated horizon Δ with past boundary at such a surface. In order for ψ to be determined uniquely by ϖ, we apply the "gauge-fixing" condition

$$\psi = 0 \quad \text{on } S_\Delta^-. \tag{4.31}$$

Once this condition is imposed, the function ψ is completely determined by the physical fields already present in the problem.

Finally, let us briefly examine the situation with the Maxwell potential $\mathbb{A}$. When constructing the phase space of Einstein–Maxwell isolated horizons, it will be most useful to have a decomposition for $\mathbb{A}$ which is analogous 4.30. Thus, we define a function χ on Δ by the conditions

$$\chi = 0 \quad \text{on } S_\Delta^-, \text{ and} \quad \mathscr{L}_\ell \chi = -\Phi_\Delta^{(\ell)}. \tag{4.32}$$

This definition is simpler than the one for ψ since we already know the leaves of the preferred foliation. Its first component gives an initial value for χ and the second describes how to "evolve" χ along the generators of Δ. Note the "evolution" equation is independent of the choice of $\ell \in [\ell]$ and the sign is chosen such that the intrinsic Maxwell potential $\underleftarrow{\mathbb{A}}$ admits a decomposition analogous to 4.30:

$$\underleftarrow{\mathbb{A}} = \mathring{\mathbb{A}} + \mathrm{d}\chi, \tag{4.33}$$

with $\ell \lrcorner \mathring{\mathbb{A}} = 0$. While $\mathring{\mathbb{A}}$ is therefore projectable to $\mathscr{P}\Delta$, unlike $\mathring{\varpi}$, its divergence there needn't vanish there and is, in fact, completely arbitrary. Thus, we have not imposed any additional gauge fixing on $\mathbb{A}$ apart from that required by condition $\mathrm{V}_{\mathrm{Max}}$.

Chapter 5

Action and Phase Space

In this chapter, we will introduce the action principle appropriate to the isolated horizon boundary conditions. We will also derive from this action the covariant phase space of histories describing isolated horizons in a given region of space-time. Finally, we will initiate the discussion of Hamiltonians describing motions along certain vector fields in these space-times and the definition of horizon mass. We will see, however, that the natural definition of horizon mass requires somewhat more care and we will postpone a more complete description for the following chapter.

5.1 ACTION PRINCIPLE

This section considers the first-order Palatini formulation of the Einstein–Hilbert action for general relativity. For technical simplicity, we reserve discussion of the Maxwell field for appendix A. The local variables describing the gravitational field in this context consist of a tetrad e_I — or, a section of the frame bundle over space-time — and a connection D in that frame bundle. Since the introduction of spinorial matter would require a tetrad, it is natural to introduce tetrads even at this early stage. The use of a first-order formalism, on the other hand, is necessary for primarily technical reasons. Specifically, the Gauss law constraint, characteristic of first-order formulations of general relativity, makes the connection between the Hamiltonian formalism and the first law of black hole mechanics particularly clear.

We begin by fixing the kinematical structure on the space of histories considered in the action principle. The specific case of interest is that of an asymptotically flat space-time $\mathcal{M}$ with an interior boundary Δ which will be an isolated horizon surface. In the case of an asymptotically flat space-time with *no* internal boundary, the kinematical structure of the space of histories is completely fixed by requiring certain asymptotic fall-off conditions on the physical fields at infinity. We will require these standard fall-off conditions in our case as well. Since boundary integrals at infinity play only a secondary role in our analysis, the precise form of the fall-off conditions will not be stated. However, we must also fix the kinematical structure at the horizon, and there we must be more explicit. We have seen in the previous chapter that every (non-extremal) isolated horizon admits a "rest frame." Since the laws of black hole mechanics concern the black hole *mass*, it is natural always to work in this rest frame for our Hamiltonian constructions. Thus, once and for all, we fix an equivalence class $[\ell]$ of vector fields on Δ and a foliation of Δ by 2-spheres S_Δ. In the space of histories for the action principle, $(\Delta, [\ell])$ is always an isolated horizon and the S_Δ are the leaves of its "rest-frame" foliation. These conditions partially restrict the diffeomorphism gauge freedom at the horizon to give us more control over the space of histories. This is analogous to the situation usually encountered at null infinity; one isolates the "universal structure" of the boundary and then fixes such a structure kinematically in order to make the action and Hamiltonian constructions simpler. In our case, it will also be convenient to fix a null tetrad $(\ell^I, n^I, m^I, \bar{m}^I)$ in the *internal* space at the horizon[1]. The frame field e_I is then constrained to satisfy $\ell^I e_I \in [\ell]$ at the horizon. In addition, we will insist the vector $n^I e_I$ be orthogonal to the leaves S_Δ of the preferred foliation of Δ.

As usual, the allowed histories in the action principle consist of smooth field configurations (e, D) satisfying the kinematical conditions stated above. Let us first recall the action appropriate to the case where the only boundary of space-time is at infinity. It is given by

$$S[e, D] = \frac{-1}{16\pi G}\int_{\mathcal{M}} \mathrm{Tr}\,[\Sigma \wedge F] + \frac{1}{16\pi G}\int_{\infty} \mathrm{Tr}\,[A \wedge \Sigma]\,, \tag{5.1}$$

where the trace is taken in the internal space and F is the curvature of the connection D. The 1-form A appearing in the last term here is the connection 1-form of D relative to a fixed, flat connection ∂ in the asymptotic region. The two-form Σ is gotten by dualizing the exterior product of a pair of co-tetrad elements on the internal indices:

$$\Sigma_I{}^J := \tfrac{1}{2}\epsilon_I{}^J{}_{KL} e^K \wedge e^L. \tag{5.2}$$

Note the surface term at infinity in 5.1. This term is needed for this action to be "differentiable," i.e., for the surface terms in its variation to vanish in the action principle. The exact form of the surface term which must be added to the bulk action is determined by the fall-off conditions imposed on the fields. We have adopted the standard fall-off conditions and thus find this standard surface term.

[1] Note that the tetrad elements m^I and $\bar{m}^I$ cannot be defined globally when the horizon topology is $S^2 \times \mathbb{R}$. Although this is an important point when considering the formulation of isolated horizons in terms of self-dual variables, it will not affect our calculations here.

With the surface term, the extrema of the action 5.1 are solutions to the usual field equations of the Palatini theory. Specifically, extremizing with respect to the connection, one obtains

$$D\Sigma = 0. \tag{5.3}$$

This equation implies the connection D is equal to the (unique) connection compatible with the frame field: $De = 0$. When this equation holds, the curvature F is related to the Riemann curvature associated with e by

$$F_{abI}{}^{J} = R_{abc}{}^{d} e^{c}_{I} e^{J}_{d}, \tag{5.4}$$

where e^I_a is the co-tetrad inverse to e^a_I: $e^a_I e^J_a = \delta^J_I$. Varying the action with respect to the frame field and using this curvature relation yields (see 5.7 below) a term

$$\mathrm{Tr}\,[\delta\Sigma \wedge F] = -2\delta e^I \wedge *G(e_I), \tag{5.5}$$

where $G(e_I)$ denotes the Einstein tensor associated with the frame field, contracted on one index with the frame vector e_I, and $*$ denotes the Hodge dual operation on space-time. In the absence of matter fields, this term must vanish. Thus, we reproduce the usual vacuum Einstein equation

$$G_{ab} = 0, \tag{5.6}$$

More generally, the additional contributions to the action from matter fields will depend on e and the right side of this equation will be proportional to the matter stress-energy.

Let us now return to the case of primary interest to us here. The space-time region $\mathcal{M}$ contemplated in the action principle has a boundary consisting of four distinct components: its future and past boundaries M^+ and M^-, its outer boundary at spatial infinity which we denote simply by ∞, and an inner boundary at an isolated horizon Δ. These boundary components are not disjoint and the intersections of $M^\pm$ with Δ and ∞ will be denoted $S^\pm_\Delta$ and $S^\pm_\infty$, respectively. These definitions are illustrated in figure 5.1. As we have discussed above, the action integral 5.1 requires a surface term at infinity in order for the surface terms at infinity in its variation to vanish and, therefore, for the extrema of 5.1 to correspond to solutions of the equations of motion. Since the space-times we consider have a new, interior, boundary component, one might expect the action 5.1 will acquire a second surface term at the horizon. Surprisingly, however, this turns out not to be the case. The variational principle for the action 5.1 is already perfectly well defined for the class of space-times under consideration. To see this, vary 5.1 to find

$$\delta(S[e,D]) = \frac{-1}{16\pi G}\int_{\mathcal{M}} \mathrm{Tr}\,[\delta\Sigma \wedge F + \delta A \wedge D\Sigma] - \frac{1}{16\pi G}\int_{\Delta\cup M^\pm} \mathrm{Tr}\,[\delta A \wedge \Sigma] \tag{5.7}$$

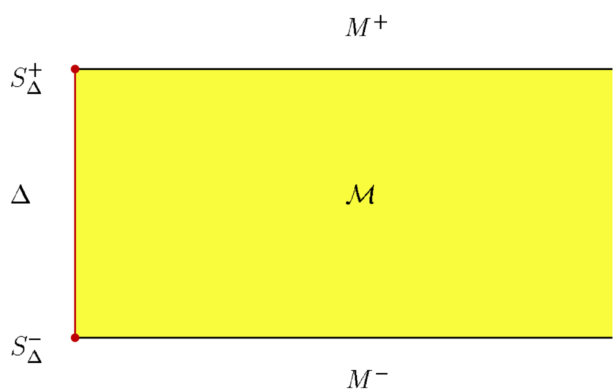

Figure 5.1: The region $\mathcal{M}$ of space-time considered in the variational principle is bounded by two partial Cauchy surfaces M^+ and M^-. They intersect the inner boundary Δ in two leaves S_Δ^+ and S_Δ^- of its preferred foliation and extend to spatial infinity. The boundary Δ itself is an isolated horizon.

where $M^{\pm}$ represents the union of M^{+} and M^{-} and the 1-form δA encodes the change in the connection D. The fields at $M^{\pm}$ are fixed in the variational principle, so we may safely drop those surface terms in this calculation. In order for the equations of motion (which are equivalent to the vanishing of the bulk term in 5.7 for generic variations) to give true extrema of the action functional, the remaining surface term at Δ must also vanish. We will now show it does.

Using the definition 4.7 of ϖ and the equations of motion at Δ, we have

$$D_{\underleftarrow{a}} \ell_I = \varpi_a \ell_I. \tag{5.8}$$

Furthermore, the frame field e and the internal tetrad $(\ell^I, n^I, m^I, \bar{m}^I)$ trivially provide a tetrad in the tangent bundle to space-time. In terms of these tetrads, one can show

$$\underleftarrow{\Sigma}^{IJ} = 2\,{}^2\!\epsilon\, \ell^{[I} n^{J]} + 2in \wedge (m\, \ell^{[I} \bar{m}^{J]} - \bar{m}\, \ell^{[I} m^{J]}). \tag{5.9}$$

The integrand of the horizon surface term in 5.7 is $\underleftarrow{\mathrm{Tr}\,[\delta A \wedge \Sigma]}$. Using 5.8 and 5.9 in that surface term, we evaluate it to be

$$\underleftarrow{\mathrm{Tr}\,[\delta A \wedge \Sigma]} = -2\delta\varpi \wedge {}^2\!\epsilon. \tag{5.10}$$

Since the fields are fixed at $M^{\pm}$ in the variational principle, this 3-form vanishes at $S^{\pm}_{\Delta}$. However, since ${}^2\!\epsilon$ is defined by pulling back the "time-independent" volume form $\hat{\epsilon}$ on $\mathscr{P}\Delta$, it must be Lie dragged along ℓ^a. Thus, using (the variation of) 4.18, we find

$$\mathscr{L}_\ell \left(\underleftarrow{\mathrm{Tr}\,[\delta A \wedge \Sigma]} \right) = -2(\mathscr{L}_\ell \delta\varpi) \wedge {}^2\!\epsilon = 2(\mathscr{L}_{\delta\ell} \varpi) \wedge {}^2\!\epsilon, \tag{5.11}$$

where we have used condition III in the second equality. Finally, since we have fixed the class $[\ell]$ of vector fields once and for all, $\delta\ell$ must be proportional to ℓ by a constant. Thus, using 4.18 again, the right side of 5.10 is Lie dragged along ℓ. Since it vanishes at $S^{\pm}_{\Delta}$, it must vanish everywhere.

We have now seen that all the surface terms in 5.7 vanish when (i) the fields are fixed at $M^{\pm}$ as in the variational principle and (ii) the isolated horizon boundary conditions hold at Δ. No additional surface term at Δ is required. The extrema of 5.1 are therefore solutions to the bulk equations of motion. An action with this property is often called "differentiable." In essence, however, this just means the variational problem for the action in question, subject to a set of known boundary conditions, is globally well-posed. There does not seem to be any strict sense in which the question of "differentiability" can be phrased.

5.2 COVARIANT PHASE SPACE

We turn now to the problem of constructing a phase space of systems containing isolated horizons. In particular, we will consider the covariant phase space framework for the space-time region $\mathcal{M}$ of the previous section. In this framework, the space on which we work consists of the solutions to the field equations in $\mathcal{M}$. The action functional gives rise to a (pre-)symplectic structure on the covariant phase space in a natural way. Note, however, that the covariant phase space is *not* generally a phase space in the usual sense since the (pre-)symplectic structure will have degenerate directions.

Let us recall the basic ingredients of the covariant phase space framework. As we mentioned previously, the space itself, denoted Ξ, consists of the extrema of the action functional under a variational principle wherein the fields are held fixed at $M^{\pm}$. Physically distinct solutions to the equations of motion, however, must have different data on $M^{\pm}$. Thus, although the action is extremized on each solution within a certain class of histories sharing its data on $M^{\pm}$, the variational principle does not apply to variations which move from one solution to another. The action functional 5.1 therefore defines a non-trivial function on the covariant phase space which we also denote by S. Given a solution $\xi \in \Xi$ to the equations of motion, a solution $\delta\xi$ to the *linearized* equations of motion on ξ defines a tangent vector to the covariant phase space. The variational formula 5.7 gives the Lie derivative of the action function along $\delta\xi$. Unlike in the previous section, however, we can always drop the bulk term in 5.7 in the covariant phase space since we always work on a solution to the equations of motion. Thus, the Lie derivative in question is always given by surface terms.

The Lie bracket of two vector fields is defined by the relation

$$[\mathscr{L}_{\delta_1\xi}, \mathscr{L}_{\delta_2\xi}]S - \mathscr{L}_{[\delta_1\xi,\delta_2\xi]}S = 0, \tag{5.12}$$

for any function S. Let us consider this relation when S is the action function on Ξ. In the variational language, the left hand side of this formula represents the anti-symmetrized second variation of the action. Since these variations are made on a solution to the equations of motion and the variations themselves solve the linearized equations of motion, the bulk term in this second variation will again vanish. Moreover, as we will see below in the isolated horizon case, the surface integrand on Δ in the second variation is an exact exterior derivative and thus may be rewritten as a pair of terms on $S_{\Delta}^{\pm}$. Therefore, the entire second variation consists of a pair of integrals over M^+ and M^-, each with additional surface terms at $S_\Delta^\pm$:

$$\delta_1(\delta_2 S) - \delta_2(\delta_1 S) - [\delta_1, \delta_2]S = \int_{M^\pm} \omega_M(\xi; \delta_1\xi, \delta_2\xi) + \oint_{S_\Delta^\pm} \omega_\Delta(\xi; \delta_1\xi, \delta_2\xi). \tag{5.13}$$

Moreover, by the general argument above, the left hand side of this equation vanishes and the integrals at M^+ and M^- give equal and opposite values. Therefore, if we reverse the orientation of one of the $M^\pm$, the integral on the right hand side will give the same value whether it is taken over M^+ or M^-. In fact, since we can repeat this argument with $\mathcal{M}$ replaced by any subset of $\mathcal{M}$, the integral

$$\Omega(\xi; \delta_1\xi, \delta_2\xi) := \int_M \omega_M(\xi; \delta_1\xi, \delta_2\xi) + \oint_{S_\Delta} \omega_\Delta(\xi; \delta_1\xi, \delta_2\xi) \tag{5.14}$$

will give the same value when integrated over *any* partial Cauchy surface in $\mathcal{M}$. In short, this means the integral in question is actually associated with the *solution* ξ and the linearizations $\delta_1\xi$ and $\delta_2\xi$ therefrom, and not with any particular slice of $\mathcal{M}$. Since Ω is anti-symmetric in $\delta_1\xi$ and $\delta_2\xi$, it defines a 2-form on Ξ. This is the (pre-)symplectic structure we seek.

The key step in the above construction of the symplectic structure lies in proving the surface term at Δ in the second variation of the action may be rewritten as an exact exterior derivative. It is not clear that this can always be done. However, although there are no concrete statements to the effect, one expects this feature might be related to the differentiability of the action discussed in the previous section. For now, we will simply focus on the action 5.1 and show it does possess this feature.

We begin with the variational formula 5.7. Taking its second variation as in the above discussion, and restricting ourselves to the covariant phase space, we find

$$\delta_1(\delta_2 S) - \delta_2(\delta_1 S) - [\delta_1, \delta_2]S = \frac{1}{16\pi G}\int_{\Delta\cup M^\pm} \mathrm{Tr}\left[\delta_1 A \wedge \delta_2\Sigma - \delta_2 A \wedge \delta_1\Sigma\right]. \tag{5.15}$$

Our task is to show the integrand on Δ can be written as an exact exterior derivative of some 2-form. First, by an argument virtually identical to that leading to 5.10, we show

$$\underleftarrow{\mathrm{Tr}\left[\delta_1 A \wedge \delta_2\Sigma\right]} = -2\delta_1\varpi \wedge \delta_2\, {}^2\!\epsilon \tag{5.16}$$

on Δ. If we decompose $\delta_1\varpi$ according to 4.30, only the term involving $\mathrm{d}\delta_1\psi$ will survive since no non-vanishing 3-form on Δ can be orthogonal to $[\ell]$. Moreover, the exterior derivative of ${}^2\!\epsilon$ is zero, so the right side of 5.16 is indeed an exact 3-form. Thus, we find the pre-symplectic structure of 5.13 is given in this case by

$$\Omega(\delta_1, \delta_2) = \frac{1}{16\pi G}\int_M \mathrm{Tr}\left[\delta_1 A \wedge \delta_2\Sigma - \delta_2 A \wedge \delta_1\Sigma\right] + \frac{1}{8\pi G}\oint_{S_\Delta} \delta_1\psi\delta_2\, {}^2\!\epsilon - \delta_2\psi\delta_1\, {}^2\!\epsilon \tag{5.17}$$

Note that the surface terms here have the opposite sign from what one might expect doing the integration by parts in 5.15. This happens because S_Δ in 5.17 is considered as the inner boundary of M, whereas in the integration by parts, $S_\Delta^\pm$ arise as the future and past boundaries of Δ. These two ways of approaching $S_\Delta^\pm$ induce opposite orientations on the 2-spheres, whence the sign change.

5.3 HORIZON DIFFEOMORPHISM GENERATORS

In the previous section, we have seen there exists a well-defined covariant phase space Ξ of asymptotically flat space-times $\mathcal{M}$ containing a single isolated horizon Δ. In this section, we consider the motions induced on Ξ by dragging the physical fields along various vector fields on $\mathcal{M}$. In particular, we are interested in whether a given infinitesimal diffeomorphism represents a canonical transformation (symplectomorpism) on phase space. Among those diffeomorphisms which are canonical transformations, we are also interested is discovering which represent physical symmetries of the theory and which are pure gauge. The analysis of this question is begun here, but will be finished in the next chapter.

To begin, not every diffeomorphism of $\mathcal{M}$ gives rise to an allowable transformation on the phase space Ξ. Only those which preserve the kinematical structure of the phase space fields should be considered. The restrictions placed on the transformations at infinity are well known, we will focus now on the restrictions arising from the kinematical structure at the horizon. Each infinitesimal diffeomorphism should preserve Δ itself, the equivalence class $[\ell]$, and the foliation of the horizon by S_Δ. The vector field generating such an infinitesimal diffeomorphism in space-time is therefore restricted to be of the form

$$W \mathrel{\hat{=}} f_{(W)}\ell + w, \tag{5.18}$$

where $f_{(W)}$ is constant over each S_Δ, $\mathscr{L}_\ell f_{(W)}$ is constant over Δ, and w is everywhere tangent to an S_Δ and satisfies $[\ell, w] = 0$. Any vector field with this structure at the horizon is allowed kinematically as a diffeomorphism of an isolated horizon space-time. We now want to analyze the question of whether these diffeomorphisms can be implemented canonically on phase space and, among those which can, which ones are physical symmetries and which are pure gauge.

Consider a vector field W on $\mathcal{M}$ satisfying 5.18. We will also allow this vector field to depend implicitly on the state of the physical fields, since we will see we must allow for this eventuality in what follows. The motion in phase space associated to the diffeomorphism along W is given simply by the Lie derivative:

$$\delta^W \Sigma = \mathscr{L}_W \Sigma = W \lrcorner D\Sigma + D(W \lrcorner \Sigma) + [\Sigma, W \lrcorner A] \tag{5.19}$$

$$\delta^W A = \mathscr{L}_W A = W \lrcorner F + D(W \lrcorner A), \tag{5.20}$$

where A is the connection 1-form of D relative to an arbitrarily chosen flat background connection ∂. One can easily verify that δ^W satisfies the linearized equations of motion and, hence, that it represents a tangent vector field on phase space. This vector field generates a canonical transformation if it preserves the symplectic structure, i.e., if $\mathscr{L}_{\delta^W}\Omega = 0$. Equivalently, δ^W is a canonical transformation if and only if there exists a Hamiltonian function H^W on phase space such that

$$\delta H^W = \Omega(\delta, \delta^W) \tag{5.21}$$

for all tangent vectors δ to phase space. As with any generally covariant theory, one expects the Hamiltonian, if it exists, will consist only of surface terms. A rather lengthy calculation in the present case reveals

$$\begin{aligned} \Omega(\delta, \delta^W) = \frac{1}{8\pi G} \oint_{S_\Delta} & \delta[(w \lrcorner \varpi)\, {}^2\!\epsilon] - (\delta w \lrcorner \varpi)\, {}^2\!\epsilon + (\mathscr{L}_W \psi - \delta^W \psi)\delta\, {}^2\!\epsilon \\ + \frac{1}{16\pi G} \oint_{S_\infty} & \mathrm{Tr}\,[\delta A \wedge (W \lrcorner \Sigma) + (W \lrcorner A)\delta\Sigma] \end{aligned} \tag{5.22}$$

The right side of this result consists of integrals both at the horizon and at infinity. Since our main concern here is with the horizon integrals, we will concentrate on the case where $W = 0$ outside some compact neighborhood of the horizon.

The first two terms of the horizon integral in 5.22 depend only on the horizontal components of W, and the third depends only on the vertical. Let us first focus on the third term which contains the difference $\mathscr{L}_W\psi - \delta^W\psi$. Since δ^W represents the dragging of the physical fields along W, one might expect this difference should vanish. However, when W has a vertical component at S_Δ^-, dragging along W will violate the "gauge-fixing" condition 4.31. On the other hand, the definition 4.30 of ψ implies $\delta^W\psi$ may differ from $\mathscr{L}_W\psi$ at most by a constant. Therefore, in order to preserve the condition 4.31, we use this constant freedom to set

$$\delta^W \psi := \mathscr{L}_W \psi - \kappa_\Delta^{(f_{(W)}^- \ell)}, \tag{5.23}$$

where $f_{(W)}^-$ is the value of $f_{(W)}$ on S_Δ^-. Since the zeroth law holds, the second, "correction" term here is indeed constant over the horizon. Furthermore, under this definition $\delta^W\psi$

vanishes at S_Δ^-, whence condition 4.31 is preserved. Now, Hamilton's equations 5.21 tell us that *if* a Hamiltonian exists which generates motions along W, it will have a surface term at the horizon whose variation is given by

$$\delta H_\Delta^W = \frac{1}{8\pi G} \oint_{S_\Delta} \delta[(w \lrcorner \varpi)\, {}^2\!\epsilon] - (\delta w \lrcorner \varpi)\, {}^2\!\epsilon + \kappa_\Delta^{(W)} \delta\, {}^2\!\epsilon, \tag{5.24}$$

where $\kappa_\Delta^{(W)}$ is a shorthand for $\kappa_\Delta^{(\bar{f}_{(W)}\ell)}$. In other words, it denotes the surface gravity associated with the vertical part of W at S_Δ^-. We can see already in this result the possible obstructions to the existence of a Hamiltonian generating motions along W. The first term in the integrand of 5.24 is an exact variation and therefore poses no problem. The other two terms, however, generally cannot be written as exact variations and therefore *can* prevent the construction of a Hamiltonian. There are a couple important exceptions, however. First, any *fixed* (i.e., state-independent) W which is horizontal everywhere on Δ will leave only the first term in the integrand of 5.24. There will exist a Hamiltonian in this case and it will be given simply by the integral being varied in that first term. Second, a purely vertical W which vanishes at S_Δ^- will also leave no non-exact terms. However, in this case, the entire right side of 5.24 vanishes, whence δ^W is a degenerate direction of the symplectic structure. In other words, these diffeomorphisms are pure gauge.

Let us conclude this discussion with a pair of remarks.

1. Any vector field W given by 5.18 can be written uniquely as a sum of two other vector fields of the same form: one with $f_{(W)}$ *constant* over Δ and the other purely vertical, with $f_{(W)}$ vanishing at S_Δ^-. Since the second vector field always generates pure gauge motions in phase space, it is natural to remove it from our consideration of the potential true symmetries of the theory. Therefore, we need only consider diffeomorphisms along vector fields of the form

$$W = c\ell + w \tag{5.25}$$

with c *constant* and w constrained as before. This fact will simplify the discussion in the next chapter considerably.

2. Among the *fixed* vector fields of the form 5.25, we also see that the only ones which generate canonical motions are those which are horizontal everywhere. To allow for diffeomorphisms with a vertical component, we *must* allow state-dependent vector fields W. This is a particularly important point since one expects time-translation symmetries, for example, should have vertical components. This point will be important to our proof of the first law of black hole mechanics.

Chapter 6

Angular Momentum, Mass and the First Law

In this chapter, we construct definitions for the mass and angular momentum of a general isolated horizon and show they are consistent with the first law of black hole mechanics. The definitions are suggested by the form of the Hamiltonians generating motions along certain vector fields. We introduce a class of axially symmetric isolated horizons for which we expect the angular momentum will be well-defined. Next, for this same class of isolated horizons, we consider the definition of horizon mass and show it requires a generalized form of the first law of black hole mechanics. Finally, we examine the first law as it appears in the isolated horizon context and make several remarks on how it can be used in variety of situations.

6.1 RIGID ROTATION AND ANGULAR MOMENTUM

To begin the discussion of angular momentum, we introduce the notion of a *rigidly rotating isolated horizon*. The intuitive idea is to define a class of isolated horizons which are analogous to the event horizons in stationary, axi-symmetric black hole space-times. Since we wish to compare these horizons to one another in a physically meaningful way, it is natural to strengthen the kinematical structure at the horizon to encode the axial symmetry. Therefore, we fix a vector field ϕ_Δ on Δ which (i) is everywhere tangent to an S_Δ, (ii) has vanishing Lie bracket with $[\ell]$, (iii) vanishes on exactly two generators of Δ, and (iv) has closed, circular orbits with affine length 2π. The first two conditions make ϕ_Δ an allowable diffeomorphism of Δ in the sense of the previous chapter and the last two give ϕ_Δ the characteristics of a *rotational* symmetry. We will now consider the phase space of isolated horizons for which this ϕ_Δ is a Killing field for the degenerate intrinsic metric on Δ. This phase space is a subset of the space we have discussed up to this point. The new restriction on the space-times we consider can be interpreted as an additional boundary condition on

e. This restriction is actually somewhat weaker than it appears. It turns out [10] that if an isolated horizon admits *any* symmetry along a vector field which is not parallel to $[\ell]$, then it must also admit an axial Killing field like ϕ_Δ. This result is very similar to Hawking's [19] and Carter's [4] proofs that a stationary, but non-static, black hole space-time must be axi-symmetric. Our new kinematical condition merely asserts that the axial symmetries "line up" when comparing rigidly rotating horizons.

Now, consider space-time vector field ϕ which approaches ϕ_Δ at Δ on each rigidly rotating isolated horizon. Such a vector field will be state-independent at Δ and will define a purely horizontal infinitesimal diffeomorphism of the horizon. The arguments of the previous chapter therefore imply δ^ϕ is a Hamiltonian vector field and the horizon surface term in the corresponding Hamiltonian is given by

$$-J_\Delta := H^\phi_\Delta = \frac{1}{8\pi G}\oint_{S_\Delta} (\phi_\Delta \lrcorner\, \varpi)\, {}^2\epsilon. \tag{6.1}$$

We *define* the angular momentum of a rigidly rotating isolated horizon to be this J_Δ. The definition is manifestly (quasi-)local to the horizon and the minus sign arises because S_Δ is an *inner* boundary of space-time. Let us now explore some of the properties of this definition.

The first question one asks is whether the event horizon of a Kerr black hole is a rigidly rotating isolated horizon and, if so, whether the above definition of angular momentum reproduces the standard result. The answer, in both cases, is in the affirmative. As discussed previously, the event horizons in the Kerr(–Newman) solutions are isolated horizons and the axi-symmetry of the ambient space-time defines the intrinsic Killing field ϕ_Δ. Thus, these are rigidly rotating isolated horizons. Now, suppose the intrinsic vector field ϕ_Δ on a rigidly rotating isolated horizon can be extended to a Killing field ϕ in a neighborhood of Δ. Consider the covector n on Δ defined by the frame field e and the fixed internal tetrad. Since n is the covariant normal to the foliation by S_Δ and is normalized such that $\ell \lrcorner\, n = -1$ everywhere, it follows that $\mathscr{L}_\ell n = 0$ and therefore that $\mathrm{d}n = 0$ on the horizon. Using this fact it is easy to show $\mathscr{D}_\ell n = \varpi$. We use this relation in the angular momentum definition 6.1 to compute

$$16\pi G\, J_\Delta = -2\oint_{S_\Delta} (\phi_\Delta \lrcorner\, \mathscr{D}_\ell n)\, {}^2\epsilon = 2\oint_{S_\Delta} (\mathscr{D}_\ell \phi_\Delta \lrcorner\, n)\, {}^2\epsilon = \oint_{S_\Delta} (\ell \lrcorner\, \mathrm{d}\phi)\cdot n\, {}^2\epsilon, \tag{6.2}$$

where we have used the Killing property of ϕ in the last equality. The integrand in the final result here can easily be rewritten as the space-time dual of $\mathrm{d}\phi$. The angular momentum of a rigidly rotating isolated horizon when a Killing extension of ϕ_Δ exists is therefore given by the Komar expression 2.17. In particular this means the angular momentum defined here will produce the standard value for a Kerr black hole.

A second question regarding the angular momentum of a rigidly rotating isolated horizon is whether it can be written in terms of the space-time curvature at Δ. Conventional wisdom has it that angular momentum is "encoded" in the Newman–Penrose component $\mathrm{Im}\,[\Psi_2]$ of the Weyl curvature. This rule of thumb is explicitly realized in the present construction. Since ϕ_Δ is a Killing vector of the intrinsic horizon geometry, it will also be a symmetry of the area element ${}^2\!\epsilon$. Thus, we find $\mathscr{L}_{\phi_\Delta}\, {}^2\!\epsilon = d(\phi_\Delta \lrcorner\, {}^2\!\epsilon) = 0$, from which it follows that the contraction $\phi_\Delta \lrcorner\, {}^2\!\epsilon$ must be the exact differential of a function f on the horizon. Now, since ϕ_Δ is tangent to S_Δ, we have

$$-8\pi G\, J_\Delta = \oint_{S_\Delta} (\phi_\Delta \lrcorner\, \varpi)\, {}^2\!\epsilon = \oint_{S_\Delta} \varpi \wedge (\phi_\Delta \lrcorner\, {}^2\!\epsilon) = \oint_{S_\Delta} 2f\, \mathrm{Im}\,[\Psi_2]\, {}^2\!\epsilon, \qquad (6.3)$$

where we have integrated by parts in the last equality and used 4.20. Thus, the angular momentum of a generic rigidly rotating isolated horizon is indeed determined by the imaginary part of Ψ_2, as expected. Note we have *not* assumed any extension of ϕ_Δ away from the horizon in obtaining this result.

We can actually carry this second result a bit further. Consider an isolated horizon which is not necessarily rigidly rotating. The construction 5.24 of the horizon surface term in the Hamiltonian indicates that a Hamiltonian will exist for *any* vector field W whose restriction to the horizon is fixed throughout phase space and is everywhere horizontal on Δ. We can define an "angular momentum" for any such vector field by evaluating the corresponding Hamiltonian surface term. When the vector field W is not a symmetry of ${}^2\!\epsilon$, the function f appearing in 6.3 does not exist. However, recall that the covector ϖ pulled back to each S_Δ is divergence-free. Since there are no harmonic 1-forms on the 2-sphere, the 1-form $\underleftarrow{\varpi}$ is therefore determined by its curl, given by 4.20. In this sense, the "angular momentum" of an isolated horizon really *is* the 2-form $2\mathrm{Im}\,[\Psi_2]\, {}^2\!\epsilon$. This identification is further supported by the definition 4.28 of non-rotating isolated horizons under which this 2-form vanishes.

6.2 DYNAMICS AND THE FIRST LAW

To state the first law of black hole mechanics for isolated horizons, we must first define a notion of horizon mass. The strategy we will use to make this definition is to introduce a time-evolution vector field t on $\mathcal{M}$ and calculate the Hamiltonian generating motions along

δ^t in phase space. That Hamiltonian will contain a surface term at Δ which is naturally identified with the energy of the horizon relative to the evolution field t. In the next section, we shall see there is a natural prescription for fixing the behavior of t at the horizon which will enable us to identify this energy with the mass.

To begin, we must specify boundary conditions on the evolution field t. At infinity, one asks that t is an asymptotic time-translation. That is, t approaches an unit time-like Killing field of the flat background metric at infinity used to define the notion of asymptotic flatness. With this choice, the surface term at infinity in the Hamiltonian H^t is equal to the ADM energy of space-time relative to the evolution field t. This energy is denoted $E^{(t)}_{\mathrm{ADM}}$. At the horizon, however, there is no fixed kinematical metric and the physical metric there needn't admit *any* Killing field. Thus, specifying the boundary condition for t at Δ is a bit more subtle. Fortunately, substantial insight can be gained by examining the situation in the stationary context. There, the stationary Killing field t is related to the axial Killing field ϕ and the horizon-generating Killing field by 2.12. Thus, the natural time-evolution vector field in the stationary context has two components at the horizon: one along the the null generator and a second, horizontal component which is proportional to an axial Killing vector. The obvious strategy is to carry this decomposition over to the broader context of isolated horizons. We have seen in the previous section that rigidly rotating isolated horizons generalize the event horizons of stationary, axi-symmetric black holes. Thus, we again consider a class of isolated horizons with a fixed, intrinsic axial symmetry ϕ_Δ. The boundary condition on t at the horizon will be that there exists a *constant* $\Omega^{(t)}_\Delta$ on Δ such that

$$t + \Omega^{(t)}_\Delta \phi_\Delta \in [\ell]. \tag{6.4}$$

This clearly mimics the structure of t in the stationary context. Note that the vertical component of t is free to vary within the equivalence class $[\ell]$. This is necessary because, as we have seen in the previous chapter, infinitesimal diffeomorphisms W with a vertical component *must* be state-dependent if the are to define Hamiltonian motions δ^W in phase space. In a more familiar terminology, this means we must allow (the boundary values of) the lapse and shift to depend on the dynamical fields. This technique of using "live" lapse and shift is routinely used in numerical relativity and in gauge-fixed calculations in canonical gravity. Also note that, although ϕ_Δ is completely fixed, the *horizontal* component of t in 6.4 is also state-dependent insofar as $\Omega^{(t)}_\Delta$ is free to depend on the dynamical fields. That is, $\Omega^{(t)}_\Delta$ must be constant on each rigidly rotating isolated horizon, but may vary from one to another. We will see shortly that the state-dependent character of t plays a critical role in the formulation of the first law of black hole mechanics for isolated horizons.

To analyze the question of whether δ^t is a Hamiltonian vector field, we must determine whether the right side of Hamilton's equations 5.21 is the exact variation of some Hamiltonian. We have already calculated the quantity in question for an arbitrary vector field W in 5.22. Moreover, setting $W = t$ and using the boundary condition that t is an asymptotic time-translation, the surface term at infinity *is* the exact variation of the ADM mass. Thus, the only potential obstruction to δ^t being Hamiltonian lies in the horizon surface term of 5.22. This surface term represents a 1-form X^t_Δ on phase space. It has been calculated in the right side of 5.24 to be

$$X^t_\Delta(\delta) = \frac{1}{8\pi G}\oint_{S_\Delta} \delta[(-\Omega^{(t)}_\Delta \phi_\Delta \lrcorner \varpi)\,{}^2\!\epsilon] - [\delta(-\Omega^{(t)}_\Delta \phi_\Delta) \lrcorner \varpi]\,{}^2\!\epsilon + \kappa^{(t)}_\Delta \delta\,{}^2\!\epsilon, \qquad (6.5)$$

where we have used the decomposition 6.4 of t into its horizontal and vertical components and $\kappa^{(t)}_\Delta$ denotes the surface gravity associated with the vertical component of t. Let us analyze each term in this result. Using the definition 6.1 of angular momentum and the constancy of $\Omega^{(t)}_\Delta$ over Δ, the first term can be written as the exact variation $\delta(\Omega^{(t)}_\Delta J_\Delta)$. In the second term, ϕ_Δ is state-independent by definition, whence δ only acts on $\Omega^{(t)}_\Delta$. Moreover, since $\Omega^{(t)}_\Delta$ is constant over Δ for each horizon, $\delta\Omega^{(t)}_\Delta$ must be so as well and the second term reduces to $(\delta\Omega^{(t)}_\Delta)J_\Delta$. The combination of the first two terms therefore leaves only $\Omega^{(t)}_\Delta \delta J_\Delta$. Finally, we use the zeroth law to calculate the last term in 6.5 and we find

$$X^t_\Delta(\delta) = \frac{\kappa^{(t)}_\Delta}{8\pi G}\,\delta A_\Delta + \Omega^{(t)}_\Delta \delta J_\Delta. \qquad (6.6)$$

The right side of this expression is remarkably similar to the first law of black hole mechanics 2.11 in the stationary context (albeit without the electric charge term which is included in the present calculation in appendix A). However, the two expressions are actually quite different in character. While 2.11 is a simple identity satisfied by the functions 2.10, the right side of 6.6 represents the contraction of the vector δ with the 1-form X^t_Δ on the phase space of rigidly rotating isolated horizons. The condition that δ^t be Hamiltonian is simply that the 1-form X^t_Δ is closed:

$$0 = \mathbb{d}X^t_\Delta = \frac{1}{8\pi G}\,\mathbb{d}\kappa^{(t)}_\Delta \wedge \mathbb{d}A_\Delta + \mathbb{d}\Omega^{(t)}_\Delta \wedge \mathbb{d}J_\Delta, \qquad (6.7)$$

where $\mathbb{d}$ denotes the exterior derivative on the (infinite-dimensional) phase space Ξ. This simple relation leads to some startling consequences for the dynamics of the horizon.

In principle, the horizon value of t depends on the state of the dynamical fields in an arbitrary way. However, if one knows the surface gravity $\kappa^{(t)}_\Delta$, the vertical component of t

is fixed uniquely, and the rotational velocity $\Omega_\Delta^{(t)}$ likewise fixes the horizontal component of t. Thus, specifying the boundary value of t on a given horizon is completely equivalent to specifying the constants $\kappa_\Delta^{(t)}$ and $\Omega_\Delta^{(t)}$. Now, the differential identity 6.7 implies the functions $\kappa_\Delta^{(t)}$ and $\Omega_\Delta^{(t)}$ on phase space can depend on the state of the dynamical fields *only* through the functions A_Δ and J_Δ. That is, the surface gravity and rotational velocity of a rigidly rotating isolated horizon are implicitly functions of its area and angular momentum. Moreover, these two functions are constrained to satisfy

$$\frac{\partial \kappa_\Delta^{(t)}(A_\Delta, J_\Delta)}{\partial J_\Delta} = 8\pi G \, \frac{\partial \Omega_\Delta^{(t)}(A_\Delta, J_\Delta)}{\partial A_\Delta} \tag{6.8}$$

by 6.7. Thus, for a Hamiltonian generating time evolution to exist, *the surface gravity and rotational velocity of a black hole must depend only upon its area and angular momentum.* Other factors, such as the *local* geometry (i.e., distortion) of the horizon, cannot affect these extrinsic parameters. Note, however, that these arguments do not indicate *which* surface gravity and rotational velocity functions one should pick. Correspondingly, for *any* choice of these functions, the Hamiltonian generating evolution along the corresponding t will exist. The horizon surface term in that Hamiltonian is a natural measure of the *energy* $E_\Delta^{(t)}$ of the horizon relative to the evolution field t. By virtue of 5.24 and the calculations above, the energy is a function only of A_Δ and J_Δ and satisfies

$$\delta E_\Delta^{(t)}(A_\Delta, J_\Delta) = \frac{\kappa_\Delta^{(t)}(A_\Delta, J_\Delta)}{8\pi G} \delta A_\Delta + \Omega_\Delta^{(t)}(A_\Delta, J_\Delta) \delta J_\Delta. \tag{6.9}$$

This relation is known as the *generalized first law of black hole mechanics.* The analogy to the usual first law 2.11 is clear. However, since there is generally no canonical choice of a single, "correct" evolution field at the horizon, there is no canonical definition for the *mass* of the isolated horizon. We will discuss how such a canonical choice of t can be made for certain classes of isolated horizons in the next section.

6.3 HORIZON MASS

We have seen that for any pair of functions $\kappa_\Delta^{(t)}(A_\Delta, J_\Delta)$ and $\Omega_\Delta^{(t)}(A_\Delta, J_\Delta)$ satisfying 6.8, the evolution field t is determined uniquely on each horizon and δ^t is Hamiltonian. It follows that the energy $E_\Delta^{(t)}(A_\Delta, J_\Delta)$ of the horizon relative to evolution along t is well-defined and satisfies the generalized first law 6.9. However, this first law holds for an *infinite* variety of evolution fields t (one for each choice of the surface gravity and rotational velocity functions) and their corresponding energies. In many physical applications, one is interested in the properties of a *specific* space-time, rather than in the structure of phase space. Then, it is useful to have at one's disposal a *single* notion of horizon energy, analogous to the ADM energy in the rest frame at infinity. This quantity could be interpreted as the horizon *mass*. In this section, we will see the horizon mass can be defined uniquely for any class of isolated horizon space-times which admit a "sufficiently complete" set of stationary examples. The general principles we will present are of a somewhat different character than most of the results in this thesis in that they are based, in large part, on conjecture with support from several specific examples. However, after the general discussion, we will analyze the definition of mass in vacuum Einstein theory in detail. These results are not based on conjecture at all; they are rigorously derived. In addition, we will discuss the case of Einstein–Maxwell isolated horizons in similar detail in appendix A.

As with an equilibrium state of a thermodynamic system, an isolated horizon is characterized by a certain set of "extrinsic parameters." Generically, these parameters include the horizon's area A_Δ, angular momentum J_Δ, surface gravity κ_Δ, rotational velocity Ω_Δ, and perhaps several other parameters describing the matter fields at the horizon. These matter field parameters can be distinguished by three properties: (i) each parameter should be preserved along the generators of Δ *as a result of boundary conditions* (i.e., without using equations of motion in the bulk of space-time), (ii) the "energy density" e in 4.12 should be expressible in terms of them, and (iii) the first law should be expressible in terms of them. This characterization is currently little more than a rule of thumb; it has been demonstrated to be sufficient in a wide variety of examples, but there exists no general argument in its support as yet. We will examine the case where there is an electro-magnetic field at the horizon in appendix A. In addition, this rule of thumb is sufficient to identify the "extrinsic parameters" of an isolated horizon in the presence of dilatonic [34, 9], Yang–Mills [35, 9] and Klein–Gordon [36] matter fields. In all these cases the general principles for the mass definition we are about to state apply.

The key idea we use to define the horizon mass is that the existence of a "sufficiently complete" set of *globally stationary* isolated horizon systems gives rise to a natural candidate for the "rest-frame" evolution field t_0 on *all* isolated horizons. Unlike a thermodynamic system, there is a natural choice of which parameters should be regarded as independent for an isolated horizon. These are the ones whose *variations* appear in the first law (area, angular momentum, etc.); all other parameters can be regarded as functions of these[1]. It

[1]There may be some other independent parameters of the horizon which are, however, *discrete*. Thus, they cannot be varied in the first law and therefore do not appear. This situation holds, for example, for

is the connection with the Hamiltonian formalism which picks out this natural set of independent parameters in the isolated horizon case. Now, the specific sense of the phrase "sufficiently complete" used above is that, for each set of values of the independent horizon parameters, there exists a *unique* stationary black hole solution. At first, this statement seems at odds with the well-known violation of the "no-hair" conjecture in, for example, Einstein–Yang–Mills theory. The conjecture states that any two black holes can be distinguished by the values of a certain collection of charges *at infinity* and there exist explicit solutions in Einstein–Yang–Mills theory which violate this conjecture . The problem arises from the topological characteristics of the Yang–Mills bundle in the bulk of space-time, leading to a discrete family of solutions which share the same asymptotic data at infinity. However, even in these cases, the relevant parameters *at the horizon* seem to be sufficient to distinguish the various stationary solutions [35, 37]. This suggests a new "no-hair" conjecture for isolated horizons — that any pair of distinct stationary black holes will have different values of the relevant isolated horizon parameters — which, if it is true, will give a proper meaning to the phrase "sufficiently complete" in the arguments made here. Let us suppose this "no-hair" conjecture holds. Then, it is natural to ask that the evolution field *t on each globally stationary solution* agrees with its stationary Killing field. With this choice of t, we can simply calculate the surface gravity, rotational velocity, and any additional parameters of the matter fields we may need on the stationary solutions. However, if the stationary solutions are in one-to-one correspondence with the values of the *independent* horizon parameters, then the *dependent* parameters of *any* isolated horizon with those same independent parameter values are determined. This happens because the first law implies the dependent parameters can only depend on the state of the system through the independent parameters. In turn, when the dependent parameters (surface gravity, rotational velocity, etc.) are fixed, then the evolution field is fixed for an *arbitrary* isolated horizon. Thus, fixing the evolution field to equal the stationary Killing field *only for the stationary solutions* actually fixes the evolution field on *every* isolated horizon. We denote this preferred evolution field by t_0. It defines a natural "rest frame" for the isolated horizon and, correspondingly, the energy $E_\Delta^{(t_0)}$ will be used to define the horizon mass.

Let us now examine the procedure outlined above in the specific case of vacuum gravity. Note that while the general argument may be somewhat vague, the procedures for the specific cases treated here and in appendix A are entirely precise. In vacuum gravity, the black hole uniqueness theorems [23] imply the only globally stationary and axi-symmetric

isolated horizons in Einstein–Yang–Mills theory [35].

solutions to Einstein's equations with regular event horizons are the Kerr black holes. These black holes are parameterized by two independent parameters which can be taken to be the area and angular momentum of the event horizon. The surface gravity and rotational velocity of the event horizon are computed in terms of these independent parameters in 2.10 (with $Q = 0$). Now, the key point is that the evolution field t is determined on *every* isolated horizon in vacuum by fixing the functions $\kappa_\Delta^{(t)}(A_\Delta, J_\Delta)$ and $\Omega_\Delta^{(t)}(A_\Delta, J_\Delta)$. If we insist the evolution field *on the Kerr solutions* agrees with the stationary Killing field with its conventional normalization at infinity, then these functions *must* be given by

$$\kappa_\Delta^{(t_0)} = \frac{R_\Delta^4 - 4G^2 J_\Delta^2)}{2R_\Delta^3 \sqrt{R_\Delta^4 + 4G^2 J_\Delta^2}} \quad \text{and} \quad \Omega_\Delta^{(t_0)} = \frac{2GJ_\Delta}{R_\Delta \sqrt{R_\Delta^4 + 4G^2 J_\Delta^2}}. \tag{6.10}$$

One can easily check these functions satisfy 6.8 as indeed they must since the first law 2.11 holds in the stationary context. This prescription fixes an evolution field t_0 on *every* isolated horizon. The energy associated with this evolution field *defines* the mass of the isolated horizon and, using the generalized first law 6.9, we can compute that mass to be

$$M_\Delta := E_\Delta^{(t_0)} = \frac{\sqrt{R^4 + 4G^2 J^2}}{2GR}. \tag{6.11}$$

Note this formula gives the same value as 2.10 does for the ADM mass of the Kerr solutions. The origin of this result is not mysterious: the generalized first law 6.9 gives M_Δ the same relation to $\kappa_\Delta^{(t_0)}$ and $\Omega_\Delta^{(t_0)}$ that the usual first law 2.11 given the ADM mass to the surface gravity and rotational velocity on the Kerr solutions. Since $\kappa_\Delta^{(t_0)}$ and $\Omega_\Delta^{(t_0)}$ take their values from the Kerr solutions, we *must* find the same mass. This fact can also be justified using a general symplectic argument [7]. Consider the Kerr solutions as points of the isolated horizon phase space. When the space-time evolution vector t_0 is a Killing vector on the stationary solutions, the phase space evolution vector δ^{t_0} may be, at most, a gauge transformation at those stationary points of phase space. The motion along δ^{t_0} is a therefore degenerate direction of the symplectic structure on the stationary points. Since this implies the right side of Hamilton's equations 5.21 will vanish, the numerical value of the Hamiltonian generating evolution along t_0 must be constant on each connected component of the submanifold of stationary points in phase space. In the Einstein(–Maxwell) case, there is only one such connected component and, moreover, the fundamental constants of the theory do not allow the construction of any quantity with units of mass. Thus, the value of the total Hamiltonian on stationary solutions must be zero:

$$H^{t_0} = H_\infty^{t_0} - H_\Delta^{t_0} = M_\Delta - M_{\text{ADM}} = 0. \tag{6.12}$$

On more general solutions, of course, M_{ADM} is greater than M_Δ. The difference is expected to equal to the energy contained in radiation in the bulk of space-time. If the the isolated horizon extends to future time-like infinity i^+ and the geometry there is sufficiently regular, one can argue (see [7] in the undistorted, non-rotating case) the difference is precisely the total energy radiated across $\mathscr{I}^+$. Hence, M_Δ generically equals the future limit of the Bondi mass. These considerations give additional support to our interpretation of M_Δ as the horizon mass.

Finally, given the canonical evolution field t_0, we can drop the superscript from the surface gravity and rotational velocity. The generalized first law 6.9 then takes the familiar form

$$\delta M_\Delta = \frac{\kappa_\Delta}{8\pi G}\,\delta A_\Delta + \Omega_\Delta\,\delta J_\Delta \tag{6.13}$$

This is the first law of black hole mechanics, adapted to (rigidly rotating) isolated horizons in vacuum Einstein theory. In contrast to the usual formulation of first law 2.11, the quantities that enter this formula are all *intrinsic* to the horizon.

Chapter 7

Discussion

Let us begin with a summary of the main ideas and results of this thesis.

In Chapter 3, we introduced the notion of an isolated horizon $(\Delta, [\ell])$. While one needs access to the entire space-time to locate an event horizon, isolated horizons can be located quasi-locally. Event horizons of stationary black holes in Einstein–Maxwell theory do qualify as isolated horizons. However, the definition does not require the presence of a Killing field even in a neighborhood of Δ. Rather, physically motivated conditions are imposed on the intrinsic geometric structure of the horizon itself. These conditions imply the Lie derivative along $[\ell]$ of both the intrinsic (degenerate) metric q and the intrinsic connection $\mathscr{D}$ vanish for generic isolated horizons. In this sense, the definition mimics the essential local structure of a Killing horizon. In addition, the area of an isolated horizon is always constant in time, whence the horizon itself is isolated or "in equilibrium." However, the space-time as a whole may admit radiation, provided only there is no flux across the horizon surface Δ. Therefore, such space-times can effectively model the late stages of a gravitational collapse. Furthermore, the quasi-local nature of the definition and the possible presence of radiation suggest the space of solutions to the Einstein–Maxwell equations admitting isolated horizons is *infinite*-dimensional. This is in striking contrast to the space of *stationary* black hole solutions which is *finite*-dimensional. Recent mathematical results by a number of workers [32, 10, 30, 31] show this expectation is indeed correct.

In chapter 4, we analyzed the structure of an isolated horizon. We were able to define surface gravity for a general isolated horizon and to show it is constant over Δ. This result extends the zeroth law of black hole mechanics from the stationary context to the much broader context of isolated horizons.

Chapters 5 and 6 were concerned with a formulation of the first law of black hole mechanics for a class of rigidly rotating isolated horizons. In order to state the law, we used Hamiltonian techniques to motivate definitions of mass and angular momentum for an isolated horizon. These definitions are viable for *any* such horizon and require neither a Killing field nor any reference to infinity. Moreover, all the definitions for the "extrinsic parameters" of an isolated horizon are entirely local to the horizon. In contrast, conventional treatments in terms of stationary black holes define some quantities at the horizon and others at infinity. For this reason, the standard arguments fail to incorporate any non-stationary situations such as one would expect to arise from a gravitational collapse. The Hamiltonian formalism gives rise to a generalized first law of black hole mechanics which relates the "extrinsic parameters" of an isolated horizon. This result sheds new light on the physical origin of the first law: it is a necessary and sufficient condition for the existence of a Hamiltonian generating time evolution on the phase space of space-times admitting isolated horizons.

Ours is not the first application of Hamiltonian techniques to the laws of black hole mechanics. In particular, Brown and York [38] and Iyer and Wald [26, 27] have used such techniques in the past, but there are important conceptual differences to our framework in both cases. While Brown and York do not restrict themselves to stationary solutions, the focus in their work is on a time-like outer boundary surrounding a space-time region containing a black hole. Our focus, of course, is on the inner boundary *at the horizon* of the black hole. The general Hamiltonian techniques we have used, particularly the covariant phase space construction, are similar to those used by Iyer and Wald. However, their formulation is tailored to the problem of perturbing stationary solutions away from stationarity. While not every such non-stationary perturbation can be modeled by an isolated horizon, the present formalism incorporates situations which are not close to stationarity in any sense.

From a geometrical perspective, the notion of an isolated horizon overlaps with that of a trapping horizon as introduced by Hayward [39]. Isolated horizons are a special case of trapping horizons, restricted primarily in that their expansion is zero. This restriction is essential to capture the notion that the horizon is in equilibrium, which in turn underlies the zeroth and first laws of black hole mechanics. Furthermore, our method of defining the surface gravity κ and the mass M_Δ of isolated horizons differ from those used by Hayward for trapping horizons and consequently our treatment of the two laws is also different. (To our knowledge, in the context of trapping horizons, a satisfactory definition of surface gravity is available only for space-times with specific global symmetries, though they needn't be stationary.) However, the notion of isolated horizon is clearly inadequate for the treatment of dynamical situations which are considered, for example, in the second law and it is these situations that provide a primary motivation in the analysis of trapping horizons.

Appendix A

Isolated Horizons in Einstein–Maxwell Theory

The purpose of this Appendix is to examine the additional terms which arise in the first law of black hole mechanics when a non-vanishing electromagnetic field is allowed at the horizon. The analysis of the boundary conditions and their consequences presented in Chapters 3 and 4 already includes the Maxwell field. Thus, we need only consider its contribution to the action and phase space formalisms of Chapter 5 and to the first law discussed in Chapter 6. Where appropriate, we will affix a subscript "G" to the quantities (action, symplectic structure, Hamiltonians, etc.) discussed in the main body of the paper to emphasize they contain contributions only from the gravitational field. Likewise, the subscript "M" will denote contributions arising from the Maxwell field.

The Maxwell field is encoded in a U(1) connection whose connection 1-form on space-time is denoted $\mathbb{A}$. We will assume $\mathbb{A}$ is globally defined on $\mathcal{M}$ (i.e., there is no magnetic charge) and therefore completely describes the Maxwell field. The gravitational field will continue to be encoded in the pair (e, D) as in the main body of the thesis. The total action for Einstein–Maxwell theory in these variables is

$$S_{\mathrm{Tot}}[e, D, \mathbb{A}] = S_{\mathrm{G}}[e, D] + S_{\mathrm{M}}[e, \mathbb{A}] \quad \text{with} \quad S_{\mathrm{M}}[e, \mathbb{A}] := \frac{1}{8\pi} \int_{\mathcal{M}} \mathbb{F} \wedge *\mathbb{F}, \tag{A.1}$$

where $S_{\mathrm{G}}[e, D]$ is given by 5.1. The dependence of the Maxwell action on the frame field e is solely due to the presence of the Hodge dual in its integrand. The equations of motion derived from this action are the same as before, but with the right side of the Einstein equation 5.6 proportional to the stress-energy 4.22 of the Maxwell field. In addition, the minimization of A.1 with respect to the Maxwell connection $\mathbb{A}$ yields the usual Maxwell equation

$$\mathrm{d} * \mathbb{F} = 0. \tag{A.2}$$

Note the Maxwell field strength is also *curl*-free because of the Bianchi identity $\mathrm{d}\mathbb{F} = 0$.

The variation of the total action acquires new terms due to the Maxwell field:

$$\delta(S_{\rm M}[e,\mathbb{A}]) = \frac{1}{8\pi}\int_{\mathcal{M}} 2\delta\mathbb{A}\wedge \mathrm{d}*\mathbb{F} + \mathbb{F}\wedge[\delta(*\mathbb{F}) - *(\delta\mathbb{F})] + \frac{1}{4\pi}\int_{\Delta\cup M^\pm}\delta\mathbb{A}\wedge *\mathbb{F}. \quad \text{(A.3)}$$

The first term of this variation gives the Maxwell equation A.2, while the second is proportional to the variation δe of the frame field. It is this term which generates the coupling of matter to curvature in the Einstein–Maxwell equations and, indeed, one finds

$$\mathbb{F}\wedge[\delta(*\mathbb{F}) - *(\delta\mathbb{F})] = -8\pi\,\delta e^I\wedge *\mathbb{T}(e_I). \quad \text{(A.4)}$$

Comparing this formula with the term 5.5 arising in the variation of the gravitational action, one finds the normalization and sign conventions are correct to give the usual Einstein–Maxwell field equation $G_{ab} = 8\pi G\mathbb{T}_{ab}$. Finally, the second integral in A.3 arises from an integration by parts in the bulk. The surface term at infinity vanishes due to standard kinematical fall-off conditions; no surface term is needed in the action A.1.

A key question for the action A.1 is whether it gives a well-posed variational problem. The surface terms at $M^\pm$ in A.3 vanish when the fields are held fixed there, but we must show the surface term at Δ vanishes due to the boundary conditions. The argument which shows it does is very similar to that used in chapter 5 to show the analogous term in the gravitational action vanishes on Δ. For the surface term in question, one has

$$\mathscr{L}_\ell\left(\underleftarrow{\delta\mathbb{A}\wedge *\mathbb{F}}\right) = -(\mathscr{L}_{\delta\ell}\underleftarrow{\mathbb{A}})\wedge\underleftarrow{*\mathbb{F}}, \quad \text{(A.5)}$$

where we have used 4.24 and (the variation of) 4.27 to simplify this expression. Since $\delta\ell$ is proportional to ℓ by a constant, the remaining Lie derivative on the right side of A.5 vanishes by 4.27. Thus, since $\delta\mathbb{A}$ vanishes at $S_\Delta^\pm$, the integrand in the surface term at Δ in A.3 vanishes everywhere. The total action A.1 is therefore "differentiable" and the variational problem is well-posed.

We now come to the construction of the covariant phase space for Einstein–Maxwell theory. The space itself again consists of the solutions to the equations of motion of $\mathcal{M}$, though these equations, of course, now are the fully coupled Einstein–Maxwell equations. The second variation 5.15 will acquire a new surface term involving the Maxwell field:

$$\delta_1(\delta_2 S_{\rm M}) - \delta_2(\delta_1 S_{\rm M}) - [\delta_1,\delta_2]S_{\rm M} = \frac{-1}{4\pi}\int_{\Delta\cup M^\pm}\delta_1\mathbb{A}\wedge\delta_2(*\mathbb{F}) - \delta_2\mathbb{A}\wedge\delta_1(*\mathbb{F}). \quad \text{(A.6)}$$

We must show the integrand for the surface integral over Δ here is once again an exact 3-form. Using the decomposition 4.33 of the Maxwell potential $\mathbb{A}$, we find

$$\underleftarrow{\delta_1\mathbb{A}\wedge\delta_2(*\mathbb{F})} = -\mathrm{d}\delta_1\chi\wedge\mathbb{E}_\Delta, \quad \text{(A.7)}$$

where $\mathbb{E}_\Delta := -\underleftarrow{*\mathbb{F}}$ denotes (lift to Δ of the) the electric flux *out* of the horizon defined in chapter 4. Since the Maxwell equations at the horizon imply the exterior derivative of this flux 2-form vanishes, the right side of A.7 *is* exact, as required. The symplectic structure is therefore well-defined in the presence of a Maxwell field. The formula 5.17 for the symplectic structure will acquire the additional term

$$\Omega_{\mathrm{M}}(\delta_1, \delta_2) = \frac{-1}{4\pi} \int_M \delta_1 \mathbb{A} \wedge \delta_2(*\mathbb{F}) - \delta_2 \mathbb{A} \wedge \delta_1(*\mathbb{F}) - \frac{1}{4\pi} \oint_{S_\Delta} \delta_1 \chi\, \delta_2 \mathbb{E}_\Delta - \delta_2 \chi\, \delta_1 \mathbb{E}_\Delta, \quad \text{(A.8)}$$

where care has been taken with the orientations of S_Δ in the surface integrals. (See the discussion following 5.17 for details.)

The kinematical diffeomorphisms of the horizon are generated by the same vector fields W in the Einstein–Maxwell case as in the vacuum case. The corresponding motion in phase space induces the same transformation on the geometric variables as before and, in addition, Lie drags the Maxwell connection:

$$\delta^W \mathbb{A} := \mathscr{L}_W \mathbb{A} = W \lrcorner\, \mathbb{F} + \mathrm{d}(W \lrcorner\, \mathbb{A}). \quad \text{(A.9)}$$

One can easily check that $\delta^W(*\mathbb{F})$ is also given by the Lie derivative of the background field. Again, we must analyze the Hamiltonian properties of this motion in phase space. Hamilton's equations in the Einstein–Maxwell case are given by 5.21, but with the symplectic structure given by the sum $\Omega_{\mathrm{Tot}}(\delta, \delta^W)$ of the Maxwell and pure gravitational pieces. Furthermore, the formula for $\delta^W \chi$ acquires a "correction" term, just as $\delta^W \psi$ does in 5.23. The point is the variation $\delta^W \chi$ cannot be identified with the Lie derivative of χ when W has a vertical component at S_Δ^- since doing so would violate the "gauge-fixing" condition 4.32. Thus, we set

$$\delta^W \chi := \mathscr{L}_W \chi + \Phi_\Delta^{(W)}, \quad \text{(A.10)}$$

where $\Phi_\Delta^{(W)}$ denotes the electric potential associated with the vertical part of W at S_Δ^-. This is directly analogous to the situation with the gravitational variables described by 5.23. The difference in sign arises because $\mathscr{L}_\ell \chi = -\Phi_\Delta^{(\ell)}$. Using A.10 in the new term in the symplectic structure gives

$$\begin{aligned}\Omega_{\mathrm{M}}(\delta_1, \delta_2) = \frac{-1}{4\pi} \oint_{S_\Delta} & \delta[(w \lrcorner\, \mathbb{A}) \mathbb{E}_\Delta] - (\delta w \lrcorner\, \mathbb{A}) \mathbb{E}_\Delta - \Phi_\Delta^{(W)} \delta \mathbb{E}_\Delta \\ + \frac{1}{4\pi} \oint_{S_\infty} & \delta \mathbb{A} \wedge (W \lrcorner *\mathbb{F}) + (W \lrcorner\, \mathbb{A}) \delta(*\mathbb{F}),\end{aligned} \quad \text{(A.11)}$$

As in the vacuum case, an allowable infinitesimal diffeomorphism W which is everywhere vertical on S_Δ and vanishes on S_Δ^- generates a gauge transformation. These diffeomorphisms may safely be excluded from the remainder of the discussion. Thus, the potential symmetries of the horizon are generated by vector fields of the form 5.25, with *constant* vertical component.

To define angular momentum for horizons with electric charge, we again focus on a class of isolated horizons which are rigidly rotating with respect to a vector field ϕ_Δ on Δ. Angular momentum is defined as the horizon surface term in the Hamiltonian generating motions along a space-time vector field ϕ which equals ϕ_Δ on Δ. The formula which replaces 6.1 in the presence of the Maxwell field is

$$-J_\Delta := H^\phi_\Delta = \frac{1}{8\pi G}\oint_{S_\Delta} (\phi_\Delta \lrcorner\, \varpi)\, {}^2\epsilon - \frac{1}{4\pi}\oint_{S_\Delta} (\phi_\Delta \lrcorner\, \mathbb{A}) E_\Delta. \tag{A.12}$$

The appearance of the second integral involving the Maxwell field, in this formula for angular momentum may seem surprising at first. Indeed the first integral, involving the gravitational degrees of freedom, reduces to the Komar expression as in the vacuum case. However, if Δ is the event horizon of a Kerr–Newman black hole, that Komar formula measures the angular momentum at the horizon. Meanwhile, 2.22 expresses the angular momentum *at infinity* in terms of that at the horizon and an additional horizon surface integral involving the Maxwell field which is of exactly the same form as the second term in A.12. Thus, J_Δ reproduces the usual angular momentum at infinity in the stationary, axi-symmetric context. However, it is manifestly (quasi-)local to the horizon and is consequently independent of any external radiative fields for generic isolated horizons.

To define mass for electrically charged isolated horizons, we repeat the argument from the vacuum case concerning the 1-form $X^t_\Delta(\delta)$, which is defined as the horizon surface term in $\Omega_{\mathrm{Tot}}(\delta, \delta^t)$. The new contribution to this surface term can be discerned from A.11. Using the modified definition A.12, we see the only new contribution to 6.6 in the Einstein–Maxwell case arises from the last term of the horizon integral in A.11. Thus, 6.6 becomes

$$X^t_\Delta(\delta) = \frac{\kappa^{(t)}_\Delta}{8\pi G}\,\delta A_\Delta + \Omega^{(t)}_\Delta \delta J_\Delta + \Phi^{(t)}_\Delta \delta \mathbb{Q}_\Delta. \tag{A.13}$$

This resembles the first law in the presence of electric charge. This relation shows the surface gravity, rotational velocity and electric potential of the horizon must all be functions of the three variables A_Δ, J_Δ and $\mathbb{Q}_\Delta$ if the evolution along t is to be Hamiltonian. Picking these functions to be given by 2.10 is the unique choice which allows t to be extended from the horizon to the stationary Killing vector in the bulk for the Kerr–Newman solutions. This choice of $t = t_0$ defines the mass of *any* isolated horizon as the energy surface term $E^{(t_0)}_\Delta$ in the associated Hamiltonian. Once again, the mass can be expressed in terms of the independent variables using the first law. The result is the same as the mass formula in 2.10.

Appendix B

The Newman–Penrose Formalism

The Newman–Penrose formalism [40] encodes the tensorial equations of general relativity in a large number of scalar equations. Doing so can often lead to simplifications in one's understanding of a given physical problem, but only at the expense of introducing a large number of new definitions. More complete reviews of the formalism than that presented here may be found in [41, 42, 43]. However, note the sign conventions used throughout this thesis agree with those of [18]. Thus, the metric signature of space-time is $(-,+,+,+)$ and the Riemann curvature tensor is defined by

$$R_{abd}{}^{d}\omega_d = (\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c \tag{B.1}$$

for any 1-form ω_c on space-time. *Both* of these conventions differ by an overall minus sign from those used in the standard references [41, 42, 43]. Thus, some of the formulae below will differ by signs from those found in the literature.

The Newman–Penrose equations are stated in terms of a tetrad $(\ell, n, m, \bar{m})$ of vectors on space-time. The vectors ℓ and n are everywhere null, while m and $\bar{m}$ are everywhere space-like and complex-conjugate to one another. Although space-time itself is a real manifold, complex vectors are used so the inner product of any tetrad vector with itself is zero. For this reason, $(\ell, n, m, \bar{m})$ is known as a *null tetrad*. The only non-vanishing inner products in the $(-,+,+,+)$ signature are $\ell \cdot n = -1$ and $m \cdot \bar{m} = +1$.

The space-time connection ∇ is encoded in the Newman–Penrose formalism by a set of twelve *spin coefficients*:

$$\begin{array}{cccc} & \ell^a\nabla m_a & \frac{1}{2}(\ell^a\nabla n_a - m^a\nabla\bar{m}_a) & \bar{m}^a\nabla n_a \\ D := \nabla_\ell & \kappa & \epsilon & \pi \\ \Delta := \nabla_n & \tau & \gamma & \nu \\ \delta := \nabla_m & \sigma & \beta & \mu \\ \bar{\delta} := \nabla_{\bar{m}} & \rho & \alpha & \lambda \end{array} \tag{B.2}$$

These definitions can also be stated implicitly as

$$
\begin{aligned}
D\ell &= (\epsilon+\bar\epsilon)\ell - \bar\kappa m - \kappa\bar m & Dn &= -(\epsilon+\bar\epsilon)n + \pi m + \bar\pi\bar m \\
\Delta\ell &= (\gamma+\bar\gamma)\ell - \bar\tau m - \tau\bar m & \Delta n &= -(\gamma+\bar\gamma)n + \nu m + \bar\nu\bar m \\
\delta\ell &= (\bar\alpha+\beta)\ell - \bar\rho m - \sigma\bar m & \delta n &= -(\bar\alpha+\beta)n + \mu m + \bar\lambda\bar m
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
Dm &= \bar\pi\ell - \kappa n + (\epsilon-\bar\epsilon)m & \Delta m &= \bar\nu\ell - \tau n + (\gamma-\bar\gamma)m \\
\delta m &= \bar\lambda\ell - \sigma n + (\beta-\bar\alpha)m & \bar\delta m &= \bar\mu\ell - \rho n + (\alpha-\bar\beta)m
\end{aligned}
$$

The spin coefficients can be interpreted geometrically as follows. The quantities ϵ, γ, α and β can be thought of as "gauge" parameters describing the parallel transport of tetrad vectors along one another. They represent the freedom available in the choice of the null tetrad from point to point. The coefficient κ measures the failure of ℓ to be geodetic, as can be seen from the first equation in B.3. Equivalently, κ describes the m and $\bar m$ components of the twist covector $\omega^{(\ell)} := *(\ell\wedge \mathrm{d}\ell)$ of ℓ. When κ vanishes, the real (imaginary) part of ρ measures the expansion (twist) of the null congruence generated by ℓ and σ measures its shear. Likewise, when ν vanishes, μ and λ measure the expansion, twist and shear of the null congruence generated by n. Finally, when κ vanishes *and* ρ is real, the quantity $\pi-(\alpha+\bar\beta)$ measures the failure of the covector $\underleftarrow{n}$ to be hypersurface-orthogonal within the null hypersurface generated by ℓ. The coefficient τ can be interpreted in a similar way, but with the roles of ℓ and n reversed. Finally, all the definitions above are invariant under the interchange

$$
\begin{matrix} \ell \rightleftharpoons n \\ m \rightleftharpoons \bar m \end{matrix} \quad \Rightarrow \quad
\begin{matrix} \epsilon \rightleftharpoons -\gamma & \alpha \rightleftharpoons -\beta \\ \kappa \rightleftharpoons -\nu & \pi \rightleftharpoons -\tau \\ \rho \rightleftharpoons -\mu & \sigma \rightleftharpoons -\lambda \end{matrix} \tag{B.4}
$$

The discussion to follow is simplified somewhat by exploiting this discrete symmetry.

Just as the space-time connection is encoded in the twelve scalar spin coefficients, its curvature is encoded in a collection of fifteen scalars known as the Newman–Penrose *curvature components*. Recall the Riemann curvature in a 4-dimensional space-time can be expressed as

$$
R_{abcd} = C_{abcd} + 2g_{[a[c}R_{d]b]} - \frac{R}{3}\, g_{a[c}g_{d]b}, \tag{B.5}
$$

where C_{abcd} is the (trace-free) Weyl tensor, R_{ab} is the Ricci tensor, and R is the scalar curvature. In terms of these tensors, the curvature components are defined as

$$\Psi_0 := C_{abcd}\ell^a m^b \ell^c m^d \qquad \Psi_1 := C_{abcd}\ell^a m^b \ell^c n^d \qquad \Psi_2 := C_{abcd}\ell^a m^b \bar{m}^c n^d$$
$$\Psi_3 := C_{abcd}\ell^a n^b \bar{m}^c n^d \qquad \Psi_4 := C_{abcd}\bar{m}^a n^b \bar{m}^c n^d$$

$$\begin{array}{lll} \Phi_{00} := \frac{1}{2}R_{ab}\ell^a\ell^b & \Phi_{01} := \frac{1}{2}R_{ab}\ell^a m^b & \Phi_{02} := \frac{1}{2}R_{ab}m^a m^b \\ \Phi_{10} := \frac{1}{2}R_{ab}\ell^a\bar{m}^b & \Phi_{11} := \frac{1}{4}R_{ab}(\ell^a n^b + m^a\bar{m}^b) & \Phi_{12} := \frac{1}{2}R_{ab}m^a n^b \\ \Phi_{20} := \frac{1}{2}R_{ab}\bar{m}^a\bar{m}^b & \Phi_{21} := \frac{1}{2}R_{ab}\bar{m}^a n^b & \Phi_{22} := \frac{1}{2}R_{ab}n^a n^b \end{array} \tag{B.6}$$

$$\Lambda := \frac{R}{24}$$

Note the Ricci components are related to one another by $\Phi_{ij} = \bar{\Phi}_{ji}$ due to the reality of the Ricci tensor. The differences in sign between our definitions of the Ricci curvature components and those found in the literature arise from the combination of our signature choice and the sign difference in the definition of the Riemann curvature.

The tetrad one uses to define the spin coefficients and connection components is not canonically fixed on space-time. The transformations which relate the different choices of tetrad can be expressed as combinations of three distinct elementary types. The first is known as a *spin-boost* transformation and is shown in table B.1. This simply rescales the null vectors ℓ and n and rotates the space-like vectors m and $\bar{m}$ into one another. The second type of elementary transformation is known as a *null rotation* about ℓ and is illustrated in table B.2. This transformation holds the null vector ℓ fixed while combining n, m and $\bar{m}$ to form a new tetrad. The third type of elementary transformation consists of a similar null rotation holding n fixed. It can be formed from the null rotation of table B.2 together with the interchange operation described by B.4.

Since the curvature is derived from the space-time connection, the curvature components must be given by derivatives of the spin coefficients. The identities which relate the two sets of variables are sometime called the Newman–Penrose "*field equations*," though they have nothing to do with the true field equations of general relativity which relate the curvature components Φ_{ij} to the matter fields. These equations break naturally into several groups. First, the optical scalars ρ and σ associated with the null vector ℓ "evolve" along ℓ according to

$$D\rho - (\epsilon + \bar{\epsilon})\rho - \bar{\delta}\kappa + (3\alpha + \bar{\beta})\kappa = \rho^2 + \sigma\bar{\sigma} + \kappa\pi - \bar{\kappa}\tau + \Phi_{00} \tag{B.9}$$

$$\ell \to \tilde{\ell} = a^2 \ell \qquad n \to \tilde{n} = a^{-2} n \qquad m \to \tilde{m} = e^{2i\theta} m$$

$$\begin{array}{lll}
\tilde{\kappa} = a^4 e^{2i\theta}\kappa & \tilde{\epsilon} = a^2[\epsilon + D(\ln a + i\theta)] & \tilde{\pi} = e^{-2i\theta}\pi \\
\tilde{\tau} = e^{2i\theta}\tau & \tilde{\gamma} = a^{-2}[\gamma + \Delta(\ln a + i\theta)] & \tilde{\nu} = a^{-4}e^{-2i\theta}\nu \\
\tilde{\sigma} = a^2 e^{4i\theta}\sigma & \tilde{\beta} = e^{2i\theta}[\beta + \delta(\ln a + i\theta)] & \tilde{\mu} = a^{-2}\mu \\
\tilde{\rho} = a^2\rho & \tilde{\alpha} = e^{-2i\theta}[\alpha + \bar{\delta}(\ln a + i\theta)] & \tilde{\lambda} = a^{-2}e^{-4i\theta}\lambda
\end{array} \tag{B.7}$$

$$\begin{array}{llll}
\tilde{\Psi}_0 = a^4 e^{4i\theta}\Psi_0 & \tilde{\Psi}_3 = a^{-2}e^{-2i\theta}\Psi_3 & \tilde{\Phi}_{00} = a^4\Phi_{00} & \tilde{\Phi}_{11} = \Phi_{11} \\
\tilde{\Psi}_1 = a^2 e^{2i\theta}\Psi_1 & \tilde{\Psi}_4 = a^{-4}e^{-4i\theta}\Psi_4 & \tilde{\Phi}_{01} = a^2 e^{2i\theta}\Phi_{01} & \tilde{\Phi}_{12} = a^{-2}e^{2i\theta}\Phi_{12} \\
\tilde{\Psi}_2 = \Psi_2 & \tilde{\Lambda} = \Lambda & \tilde{\Phi}_{02} = e^{4i\theta}\Phi_{02} & \tilde{\Phi}_{22} = a^{-4}\Phi_{22}
\end{array}$$

Table B.1: A spin-boost transformation with real parameters a and θ.

$$\ell \to \hat{\ell} = \ell \qquad m \to \hat{m} = m + \bar{c}\ell \qquad n \to \hat{n} = n + cm + \bar{c}\bar{m} + c\bar{c}\ell$$

$$\begin{array}{llll}
\hat{\kappa} = \kappa & \hat{\tau} = \tau + c\sigma + \bar{c}\rho + c\bar{c}\kappa & \hat{\gamma} = \gamma + \bar{c}\alpha + c(\tau+\beta) + c\bar{c}(\rho+\epsilon) + c^2\sigma + c^2\bar{c}\kappa \\
\hat{\epsilon} = \epsilon + c\kappa & \hat{\alpha} = \alpha + c\epsilon + c\rho + c^2\kappa & \hat{\lambda} = \lambda + c\pi + 2c\alpha + c^2(\rho + 2\epsilon) + c^3\kappa + cDc + \bar{\delta}c \\
\hat{\sigma} = \sigma + \bar{c}\kappa & \hat{\beta} = \beta + c\sigma + \bar{c}\epsilon + c\bar{c}\kappa & \hat{\mu} = \mu + 2c\beta + \bar{c}\pi + c^2\sigma + 2c\bar{c}\epsilon + c^2\bar{c}\kappa + \bar{c}Dc + \delta c \\
\hat{\rho} = \rho + c\kappa & \hat{\pi} = \pi + 2c\epsilon + c^2\kappa + Dc & \hat{\nu} = \nu + c(2\gamma + \mu) + \bar{c}\lambda + c^2(\tau + 2\beta) + c\bar{c}(\pi + 2\alpha) + c^3\sigma \\
 & & \qquad + c^2\bar{c}(\rho + 2\epsilon) + c^3\bar{c}\kappa + \Delta c + c\delta c + \bar{c}\bar{\delta}c + c\bar{c}Dc
\end{array} \tag{B.8}$$

$$\begin{array}{ll}
\hat{\Lambda} = \Lambda & \hat{\Phi}_{00} = \Phi_{00} \\
\hat{\Psi}_0 = \Psi_0 & \hat{\Phi}_{01} = \Phi_{01} + \bar{c}\Phi_{00} \\
\hat{\Psi}_1 = \Psi_1 + c\Psi_0 & \hat{\Phi}_{02} = \Phi_{02} + 2\bar{c}\Phi_{01} + \bar{c}^2\Phi_{00} \\
\hat{\Psi}_2 = \Psi_2 + 2c\Psi_1 + c^2\Psi_0 & \hat{\Phi}_{11} = \Phi_{11} + c\Phi_{01} + \bar{c}\Phi_{10} + c\bar{c}\Phi_{00} \\
\hat{\Psi}_3 = \Psi_3 + 3c\Psi_2 + 3c^2\Psi_1 + c^3\Psi_0 & \hat{\Phi}_{12} = \Phi_{12} + c\Phi_{02} + 2\bar{c}\Phi_{11} + 2c\bar{c}\Phi_{01} + \bar{c}^2\Phi_{10} + c\bar{c}^2\Phi_{00} \\
\hat{\Psi}_4 = \Psi_4 + 4c\Psi_3 + 6c^2\Psi_2 + 4c^3\Psi_1 + c^4\Psi_0 & \hat{\Phi}_{22} = \Phi_{22} + 2c\Phi_{12} + 2\bar{c}\Phi_{21} + c^2\Phi_{02} + 2c\bar{c}\Phi_{11} + \bar{c}^2\Phi_{20} \\
 & \qquad + 2c^2\bar{c}\Phi_{01} + 2c\bar{c}^2\Phi_{10} + c^2\bar{c}^2\Phi_{00}
\end{array}$$

Table B.2: A null rotation about ℓ with complex parameter c.

$$D\sigma - (3\epsilon - \bar{\epsilon})\sigma - \delta\kappa + (3\beta + \bar{\alpha})\kappa = (\rho + \bar{\rho})\sigma - (\tau - \bar{\pi})\kappa + \Psi_0. \tag{B.10}$$

When $\kappa = 0$ and ℓ is geodetic, these are identical to the Raychaudhuri equations 4.5 and 4.6, together with a third equation describing the "evolution" of the twist of ℓ. Second, the optical scalars μ and λ associated with n "evolve" along ℓ according to

$$D\mu + (\epsilon + \bar{\epsilon})\mu - \delta\pi - (\beta - \bar{\alpha})\pi = \pi\bar{\pi} - \kappa\nu + \bar{\rho}\mu + \sigma\lambda + \Psi_2 + 2\Lambda \tag{B.11}$$

$$D\lambda + (3\epsilon - \bar{\epsilon})\lambda - \bar{\delta}\pi - (\alpha - \bar{\beta})\pi = \pi^2 - \bar{\kappa}\nu + \rho\lambda + \bar{\sigma}\mu + \Phi_{20}. \tag{B.12}$$

Third, in addition to these evolution equations, there are "constraint" equations which entangle the two pairs of optical scalars. These equations link the derivatives of all four quantities in the space-like directions normal to both ℓ and n:

$$\delta\rho - (\beta + \bar{\alpha})\rho - \bar{\delta}\sigma + (3\alpha - \bar{\beta})\sigma = (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01} \tag{B.13}$$

$$\delta\lambda + (3\beta - \bar{\alpha})\lambda - \bar{\delta}\mu - (\alpha + \bar{\beta})\mu = (\mu - \bar{\mu})\pi + (\rho - \bar{\rho})\nu - \Psi_3 + \Phi_{21}. \tag{B.14}$$

Fourth, there are formulae which describe the curvature of the connection associated with the gauge freedom in the choice of tetrad. Pulling these equations back into the 3-planes orthogonal to ℓ yields

$$D\alpha + (\epsilon - \bar{\epsilon})\alpha - \bar{\delta}\epsilon + (\alpha + \bar{\beta})\epsilon - \epsilon\pi + \gamma\bar{\kappa} - \beta\bar{\sigma} - \alpha\rho = \rho\pi - \kappa\lambda + \Phi_{10} \tag{B.15}$$

$$D\beta - (\epsilon - \bar{\epsilon})\beta - \delta\epsilon + (\beta + \bar{\alpha})\epsilon - \epsilon\bar{\pi} + \gamma\kappa - \beta\bar{\rho} - \alpha\sigma = \sigma\pi - \kappa\mu + \Psi_1 \tag{B.16}$$

$$\delta\alpha + (\beta - \bar{\alpha})\alpha - \bar{\delta}\beta + (\alpha - \bar{\beta})\beta - \epsilon(\mu - \bar{\mu}) - \gamma(\rho - \bar{\rho}) = \rho\mu - \lambda\sigma - \Psi_2 + \Phi_{11} + \Lambda \tag{B.17}$$

All of the "field equations" we have stated thus far involve derivatives only along the 3-planes orthogonal to ℓ. One can formulate an analogous set of equations involving derivatives in the 3-planes orthogonal to n. Although this second set of equations are not particularly useful in this thesis, we will state them for completeness. The "evolution" equations along n are

$$\Delta\mu + (\gamma + \bar{\gamma})\mu - \delta\nu - (3\beta + \bar{\alpha})\nu = -\mu^2 - \lambda\bar{\lambda} - \nu\tau + \bar{\nu}\pi - \Phi_{22} \tag{B.18}$$

$$\Delta\lambda + (3\gamma - \bar{\gamma})\lambda - \bar{\delta}\nu - (3\alpha + \bar{\beta})\nu = -(\mu + \bar{\mu})\lambda + (\pi - \bar{\tau})\nu - \Psi_4 \tag{B.19}$$

$$\Delta\rho - (\gamma + \bar{\gamma})\rho - \bar{\delta}\tau + (\alpha - \bar{\beta})\tau = -\tau\bar{\tau} + \nu\kappa - \bar{\mu}\rho - \lambda\sigma - \Psi_2 - 2\Lambda \tag{B.20}$$

$$\Delta\sigma - (3\gamma - \bar{\gamma})\sigma - \delta\tau + (\beta - \bar{\alpha})\tau = -\tau^2 + \bar{\nu}\kappa - \mu\sigma - \bar{\lambda}\rho - \Phi_{02}. \tag{B.21}$$

Since the congruences generated by ℓ^a and n^a intersect in surfaces which are transverse to both vectors, the "constraint" equations will be the same as before. However, there are also a set of equations which relate derivatives of the spin-coefficients in the 2-planes tangent to both ℓ and n:

$$D\tau - (\epsilon - \bar{\epsilon})\tau - \Delta\kappa + (3\gamma + \bar{\gamma})\kappa = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + \Psi_1 + \Phi_{01} \tag{B.22}$$

$$D\nu + (3\epsilon + \bar{\epsilon})\nu - \Delta\pi - (\gamma - \bar{\gamma})\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + \Psi_3 + \Phi_{21}. \tag{B.23}$$

Finally, there are components of the "gauge" curvature equations along the 3-planes orthogonal to n and in the 2-planes tangent to both ℓ and n:

$$\Delta\beta - (\gamma - \bar{\gamma})\beta - \delta\gamma - (\beta + \bar{\alpha})\gamma + \gamma\tau - \epsilon\bar{\nu} + \alpha\bar{\lambda} + \beta\mu = -\mu\tau + \sigma\nu - \Phi_{12} \tag{B.24}$$

$$\Delta\alpha + (\gamma - \bar{\gamma})\alpha - \bar{\delta}\gamma - (\alpha + \bar{\beta})\gamma + \gamma\bar{\tau} - \epsilon\nu + \alpha\bar{\mu} + \beta\lambda = -\lambda\tau + \nu\rho - \Psi_3 \tag{B.25}$$

$$D\gamma + (\epsilon + \bar{\epsilon})\gamma - \Delta\epsilon + (\gamma + \bar{\gamma})\epsilon - \beta(\bar{\tau} + \pi) - \alpha(\tau + \bar{\pi}) = \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \Lambda \tag{B.26}$$

These eighteen equations allow one to compute the space-time curvature from the spin coefficients associated with a given tetrad. They are somewhat over-complete in the sense there are eighteen complex equations to determine only eight complex (Ψ_i and Φ_{ij} with $j > i$) and four real (Φ_{ii} and Λ) unknowns. The redundancy, however, allows one some freedom to choose the most convenient subset of equations.

The last set of equations in the Newman–Penrose formalism encode the Bianchi identities satisfied by the space-time curvature. These necessarily involve derivatives of the curvature components. The equations stated here are in a slightly different form from the usual presentations, but these are better suited to application to isolated horizons. The Bianchi identities involving derivatives only along the 3-planes orthogonal to ℓ are

$$\begin{aligned} &D(\Psi_1 - \Phi_{01}) - 2\epsilon(\Psi_1 - \Phi_{01}) + \delta\Phi_{00} - 2(\bar{\alpha} + \beta)\Phi_{00} - \bar{\delta}\Psi_0 + 4\alpha\Psi_0 \\ &\quad = \pi\Psi_0 + 4\rho\Psi_1 - \kappa(3\Psi_2 - 2\Phi_{11}) - 2\bar{\rho}\Phi_{01} - 2\sigma\Phi_{10} + \bar{\kappa}\Phi_{02} - \bar{\pi}\Phi_{00} \end{aligned} \tag{B.27}$$

$$\begin{aligned} &D(\Psi_2 - \Phi_{11} - \Lambda) + \delta\Phi_{10} - 2\bar{\alpha}\Phi_{10} - \bar{\delta}\Psi_1 + 2\alpha\Psi_1 \\ &\quad = -\lambda\Psi_0 + 2\pi\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 + \mu\Phi_{00} - 2\bar{\rho}\Phi_{11} - \sigma\Phi_{20} + 2\mathrm{Re}\left[\kappa\Phi_{21} - \pi\Phi_{01}\right] \end{aligned} \tag{B.28}$$

$$\begin{aligned} &D(\Psi_3 - \Phi_{21}) + 2\epsilon(\Psi_3 - \Phi_{21}) + \delta\Phi_{20} + 2(\beta - \bar{\alpha})\Phi_{20} - \bar{\delta}(\Psi_2 + 2\Lambda) \\ &\quad = -2\lambda\Psi_1 + \pi(3\Psi_2 - 2\Phi_{11}) + 2\rho\Psi_3 - \kappa\Psi_4 + 2\mu\Phi_{10} - \bar{\pi}\Phi_{20} - 2\bar{\rho}\Phi_{21} + \bar{\kappa}\Phi_{22}. \end{aligned} \tag{B.29}$$

These are the Bianchi identities which are most useful in the isolated horizon context. There is a second set of equations which involve only derivatives in the 3-planes orthogonal to n.

These include

$$\begin{aligned}&\Delta(\Psi_1-\Phi_{01})-2\gamma(\Psi_1-\Phi_{01})-\delta(\Psi_2+2\Lambda)+\bar{\delta}\Phi_{02}-2(\alpha-\bar{\beta})\Phi_{02}\\&\quad=\nu\Psi_0-2\mu\Psi_1-\tau(3\Psi_2-2\Phi_{1}1)+2\sigma\Psi_3-\bar{\nu}\Phi_{00}+2\bar{\mu}\Phi_{01}+\bar{\tau}\Phi_{02}-2\rho\Phi_{12}\end{aligned} \tag{B.30}$$

$$\begin{aligned}&\Delta(\Psi_2-\Phi_{11}-\Lambda)-\delta\Psi_3-2\beta\Psi_3+\bar{\delta}\Phi_{12}+2\bar{\beta}\Phi_{12}\\&\quad=2\nu\Psi_1-3\mu\Psi_2-2\tau\Psi_3+\sigma\Psi_4-2\bar{\mu}\Phi_{11}+\lambda\Phi_{02}-\rho\Phi_{22}+2\mathrm{Re}\left[\tau\Phi_{21}-\nu\Phi_{01}\right]\end{aligned} \tag{B.31}$$

$$\begin{aligned}&\Delta(\Psi_3-\Phi_{21})+2\gamma(\Psi_3-\Phi_{21})-\delta\Psi_4-4\beta\Psi_4+\bar{\delta}\Phi_{22}+2(\alpha+\bar{\beta})\Phi_{22}\\&\quad=\nu(3\Psi_2-2\Phi_{11}-4\mu\Psi_3-\tau\Psi_4-\bar{\nu}\Phi_{20}+2\lambda\Phi_{12}+2\bar{\mu}\Phi_{21}+\bar{\tau}\Phi_{2}2\end{aligned} \tag{B.32}$$

Finally, there are several Bianchi identities which involve derivatives along both ℓ and n. These are

$$\begin{aligned}&D\Psi_4+4\epsilon\Psi_4+\Delta\Phi_{20}+2(\gamma-\bar{\gamma})\Phi_{20}-\bar{\delta}(\Psi_3+\Phi_{21})-2\alpha(\Psi_3+\Phi_{21})\\&\quad=-\lambda(3\Psi_2+\Phi_{11})+4\pi\Psi_3+\rho\Psi_4+2\nu\Phi_{10}-\bar{\mu}\Phi_{20}-2\bar{\tau}\Phi_{21}+\bar{\sigma}\Phi_{22}\end{aligned} \tag{B.33}$$

$$\begin{aligned}&\Delta\Psi_0-4\gamma\Psi_0+D\Phi_{02}-2(\epsilon-\bar{\epsilon})\Phi_{02}-\delta(\Psi_1+\Phi_{01})+2\beta(\Psi_1+\Phi_{01})\\&\quad=-\mu\Psi_0-4\tau\Psi_1+\sigma(3\Psi_2+2\Phi_{11})-\bar{\lambda}\Phi_{00}+2\bar{\pi}\Phi_{01}+\bar{\rho}\Phi_{02}-2\kappa\Phi_{12}\end{aligned} \tag{B.34}$$

$$\begin{aligned}&D(\Psi_2+2\Lambda)+\Delta\Phi_{00}-2(\gamma+\bar{\gamma})\Phi_{00}-\bar{\delta}(\Psi_1+\Phi_{01})+2\alpha(\Psi_1+\Phi_{01})\\&\quad=-\lambda\Psi_0+2\pi\Psi_1+\rho(3\Psi_2+2\Phi_{11})-2\kappa\Psi_3-\bar{\mu}\Phi_{00}-2\bar{\tau}\Phi_{01}-2\tau\Phi_{10}+\bar{\sigma}\Phi_{02}\end{aligned} \tag{B.35}$$

$$\begin{aligned}&\Delta(\Psi_2+2\Lambda)+D\Phi_{22}+2(\epsilon+\bar{\epsilon})\Phi_{22}-\delta(\Psi_3+\Phi_{21})-2\beta(\Psi_3+\Phi_{21})\\&\quad=2\nu\Psi_1-\mu(3\Psi_2+2\Phi_{11})-2\tau\Psi_3+\sigma\Psi_4-\bar{\lambda}\Phi_{20}+2\pi\Phi_{12}+2\bar{\pi}\Phi_{21}+\bar{\rho}\Phi_{22}\end{aligned} \tag{B.36}$$

$$\begin{aligned}&D\Phi_{12}+2\bar{\epsilon}\Phi_{12}+\Delta\Phi_{01}-2\gamma\Phi_{01}-\delta(\Phi_{11}-3\Lambda)-\bar{\delta}\Phi_{02}+2(\alpha-\bar{\beta})\Phi_{02}\\&\quad=\bar{\nu}\Phi_{00}-(\mu+2\bar{\mu})\Phi_{01}-\bar{\lambda}\Phi_{10}+2(\bar{\pi}-\tau)\Phi_{11}\\&\qquad+(\pi-\bar{\tau})\Phi_{02}+(2\rho+\bar{\rho})\Phi_{12}+\sigma\Phi_{21}-\kappa\Phi_{22}.\end{aligned} \tag{B.37}$$

This concludes the presentation of the Newman–Penrose formalism.

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Vita

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